5. Note on the Ideal Class Group of Abelian Number Fields

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For any algebraic number field k, $C(k)$ will denote the ideal class group of k. For any abelian group G and an integer m, G^m will mean the subgroup of G consisting of m-th powers of element of G.

The purpose of this note is to prove:

Theorem. Let L be an abelian number field and K a subfield of L of degree n. Then $C(L)$ contains a subgroup which is isomorphic to $C(K)^n$.

Proof. Let $\tilde{L}$ be the Hilbert class field of the field L and $\tilde{K}$ be the Hilbert class field of the field K. By Galois theory, we have the following exact sequence

$$\mathrm{Gal}(\tilde{L}/L) \to \mathrm{Gal}(\tilde{K}/K) \to \mathrm{Gal}(\tilde{K} \cap L/K) \to 0.$$

By class field theory, this gives us the exact sequence

$$C(L)^{N_{L/K}} \to C(K)^f \to \mathrm{Gal}(\tilde{K} \cap L/K) \to 0.$$

This implies our Theorem owing to the following Lemma.

Lemma. We have $C(K)^n \subset N_{L/K}(C(L))$.

Proof. From now on, we will write the occuring class groups additively. Let $x \in C(K)$. Since $C(Q) = 0$, we have that $\sum_{\sigma \in G} \sigma \cdot x = 0$, where $G = \mathrm{Gal}(K/Q)$. Therefore $nx = nx - \sum_{\sigma \in G} \sigma \cdot x = \sum_{\sigma \in G} (1 - \sigma)x$. Since $\tilde{K} \cap L$ is abelian over Q, the group G acts trivially on $\mathrm{Gal}(\tilde{K} \cap L/K)$. Therefore the G-homomorphism f maps each $(1 - \sigma)x$ to 0 and we see that $f(nx) = 0$ which, by exactness, implies that $nx \in$ image $C(L)$ as required. This completes the proof.

Using this lemma we have clearly that $C(L)$ contains a subgroup isomorphic to $C(K)^n$. This completes the proof.

Remark. Our Theorem generalizes the main theorem of [1].

References

[1] H. Osada: Note on the class-number of the maximal real subgroup of a cyclotomic field. II. Nagoya Math. J., **113**, 147–151 (1989).

[2] L. Washington: Introduction to cyclotomic field. Graduate Texts in Math., **83**, Springer, Berlin, Heidelberg, New York (1982).