24. *Deformation Quantization of Poisson Algebras*

By Hideki OMORI,*) Yoshiaki MAEDA,**) and Akira YOSHIOKA*)

(Communicated by Kunihiko KODAIRA, M. J. A., May 12, 1992)

§0. **Introduction.** Let M be a C^∞ Poisson manifold, and $C^\infty(M)$ the set of all $\boldsymbol{C}$-valued C^∞ functions on M. In what follows, we put $\mathfrak{a}=C^\infty(M)$ for simplicity. By definition of Poisson manifolds, there exists a bilinear map $\{\ ,\ \}: \mathfrak{a}\times\mathfrak{a}\to\mathfrak{a}$, called the *Poisson bracket*, with the following properties: For any $f, g, h\in\mathfrak{a}$,

$$\begin{cases}\{f,g\}=-\{g,f\}, \quad \{f,g\cdot h\}=\{f,g\}\cdot h+g\cdot\{f,h\},\\ \{f,\{g,h\}\}+\{g,\{h,f\}\}+\{h,\{f,g\}\}=0.\end{cases}$$

The algebra $(\mathfrak{a},\{\ ,\ \})$ is called the *Poisson algebra*.

We introduce the notion of the *deformation quantization* for the Poisson algebra $(\mathfrak{a},\{\ ,\ \})$ as follows: Let $\mathfrak{a}[\![\nu]\!]$ be the direct product space $\prod_{m=0}^{\infty}\nu^m\mathfrak{a}$, where ν is a formal parameter. Consider an associative product $*: \mathfrak{a}[\![\nu]\!]\times\mathfrak{a}[\![\nu]\!]\to\mathfrak{a}[\![\nu]\!]$ such that ν is a center of $(\mathfrak{a}[\![\nu]\!], *)$ and 1 is the identity. Set for any $f, g\in\mathfrak{a}$, $f*g=\sum_{n=0}^{\infty}\nu^n\pi_n(f,g)$ according to the decomposition of $f*g$.

Let $\hat{\mathfrak{a}}_k=\mathfrak{a}[\![\nu]\!]/\nu^{k+1}\mathfrak{a}[\![\nu]\!]\equiv\mathfrak{a}\oplus\nu\mathfrak{a}\oplus\cdots\oplus\nu^k\mathfrak{a}$. Then an associative product $*$ can be considered on $\hat{\mathfrak{a}}_k$ by setting $\nu^{k+1}=0$. We denote this algebra by $(\hat{\mathfrak{a}}_k, \{\pi_m\}_{m=0}^{k})$.

Definition. (i) For $k\geq 2$, an associative algebra $(\hat{\mathfrak{a}}_k, \{\pi_m\}_{m=0}^{k})$ is called a *deformation quantization of order* k of $(\mathfrak{a},\{\ ,\ \})$, if the following conditions are satisfied: For any $f, g\in\mathfrak{a}$.

(a) $\pi_0(f,g)=f\cdot g$ and $\pi_1(f,g)=-\frac{1}{2}\{f,g\}$.

(b) π_m is a bidifferential operator and $\pi_m(f,g)=(-1)^m\pi_m(g,f)$, $0\leq m\leq k$.

(ii) $(\mathfrak{a}[\![\nu]\!], *)$ is called a *deformation quantization* of $(\mathfrak{a},\{\ ,\ \})$, if $(\hat{\mathfrak{a}}_k, \{\pi_m\}_{m=0}^{k})$ is a deformation quantization of order k of $(\mathfrak{a},\{\ ,\ \})$ for any k.

The main purpose of this paper is to study the obstructions for a deformation quantization $(\hat{\mathfrak{a}}_k, \{\pi_m\}_{m=0}^{k})$ of order k to be extended to that of order $k+1$. The obstruction R_{k+1} is obtained as a deRham-Chevally 3-cocycle defined in the next section.

Our main theorem is stated as follows:

Main theorem. *Suppose* $(\hat{\mathfrak{a}}_k, \{\pi_m\}_{m=0}^{k})$ *is a deformation quantization of order* k *of* $(\mathfrak{a},\{\ ,\ \})$. *There exists a deformation quantization* $(\hat{\mathfrak{a}}_{k+1}, \{\pi_m\}_{m=0}^{k+1})$ *of order* $k+1$ *if and only if* $R_{k+1}=0$.

---

*) Department of Mathematics, Faculty of Science and Technology, Science University of Tokyo.

**) Department of Mathematics, Faculty of Science and Technology, Keio University.