78. *A Criterion for Multivalent Functions*

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Abstract: A more general criterion for multivalent functions is obtained. The result of this paper is the extension of the former results of Ozaki [1], Nunokawa [2], Nunokawa and Hoshino [3].

1. Introduction. It is well-known that if a function $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ is analytic and satisfies the condition $\operatorname{Re} f'(z) > 0$ in the unit disk $E = \{z : |z| < 1\}$, then $f(z)$ is univalent in E. Ozaki [1, Theorem 2] extended this result to the following:

If $f(z)$ is analytic in a convex domain D and $\operatorname{Re}(e^{i\alpha} f^{(p)}(z)) > 0$ in D, where α is a real constant, then $f(z)$ is at most p-valent in D.

This shows that if $f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n$ is analytic in E and $\operatorname{Re}(f^{(p)}(z)) > 0$ in E, then $f(z)$ is p-valent in E.

The above result was improved as follows:

Theorem A ([2]). *Let $p \geqslant 2$. If $f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n$ is analytic in E and*

$$(1,1) \qquad \operatorname{Re} f^{(p)}(z) > -\frac{\log^{(4/e)}}{2\log(e/2)} p! \text{ in } E,$$

then $f(z)$ is p-valent in E.

Theorem B ([3]). *Let $p \geqslant 3$. If $f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n$ is analytic in E and*

$$(1,2) \qquad \operatorname{Re} f^{(p)}(z) > -\frac{1 - 4\log(4/e)\log(e/2)}{4\log(4/e)\log(e/2)} p! \text{ in } E,$$

then $f(z)$ is p-valent in E.

In the present paper, we shall give a more general theorem which extends the above results.

2. Main Result. In order to derive our main result, we need the following lemmata.

Lemma 1 ([3]). *Let $p(z)$ be analytic in E with $p(0) = 1$. Suppose that $\alpha > 0$, $\beta < 1$ and that for $z \in E$, $\operatorname{Re}(p(z) + \alpha z p'(z)) > \beta$. Then for $z \in E$,*

$$(2,1) \qquad \operatorname{Re}(p(z)) > 1 + 2(1-\beta) \sum_{n=1}^{\infty} \frac{(-1)^n}{1+\alpha n}.$$

The estimate is best possible for

$$(2,2) \qquad p_o(z) = 1 + 2(1-\beta) \sum_{n=1}^{\infty} \frac{(-1)^n}{1+\alpha n} z^n.$$

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