69. *Improvement in the Irrationality Measures of π and π^2*

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(Communicated by Shokichi IYANAGA, M. J. A., Nov. 12, 1992)

§0. In this note we show the new results concerning lower bounds for rational approximations to π, π^2 and some other numbers involving π. These bounds will be derived from particular integrals of some rational functions involving Legendre type polynomials.

W. M. Schmidt [18] stated that a Roth type theorem should hold for the classical constants in analysis such as π, π^2, log 2, $\zeta(3)$, $\cdots$, as Lang had said that it should hold for any "reasonably" defined number. Note that it does hold for almost all transcendental numbers. Our results in this note are, however, still far from the conjecture but these effective results can be considered as a step to this direction.

§1. The following lemma due to F. Beukers [2] is very important in the study of rational approximations to $\zeta(2)=\pi^2/6$. Let D_n be the least common multiple of $\{1,2,\cdots,n\}$. For any polynomial $P(x)$ let $\deg(P)$ and $\mathrm{ord}(P)$ be the degree and the order of zero point at the origin of $P(x)$ respectively. Put $S=[0,1]\times[0,1]$.

Lemma 1.1. *For any polynomials $f(z)$ and $g(z)$ with integral coefficients, we have*

$$\int_S\int \frac{f(x)g(y)}{1-xy}\,dx\,dy = a\,\zeta(2)+b,$$

where

$$a=\frac{1}{2\pi i}\int_C f(z)g\left(\frac{1}{z}\right)\frac{dz}{z}$$

is an integer (C denotes a closed curve enclosing the origin) and b is a rational number whose denominator is a divisor of D_ND_M with $M=\max\{\deg(f),\deg(g)\}$ and

$$N=\min\{\max\{\deg(f),\deg(g)-\mathrm{ord}(f)\},\ \max\{\deg(g),\deg(f)-\mathrm{ord}(g)\}\}.$$

We now consider the following double integral:

$$\varepsilon_n=\int_S\int\frac{(x(1-x))^{15n}(y(1-y))^{14n}}{(1-xy)^{12n+1}}\,dx\,dy \tag{1}$$

for any integer $n\geq 1$. After k-fold and $(12n-k)$-fold partial integrations with respect to x and y respectively, it follows that

$$\binom{12n}{k}\varepsilon_n=\int_S\int\frac{F_k(x)G_k(y)}{1-xy}\,dx\,dy \tag{2}$$

for any $k\in[0,12n]$, where