68. *On the Pro-p Gottlieb Theorem*

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The purpose of this note is to present a remark on center-triviality of certain pro-p groups. We shall show the following

Theorem 1. *Let p be a rational prime, G a pro-p group, and $\boldsymbol{F}_p$ the trivial G-module of order p. Suppose that the following three conditions are satisfied.*

(1) $$cd_p G = n < \infty,$$

(2) $$H^i(G, \boldsymbol{F}_p) \text{ is finite for } i \geq 0,$$

(3) $$\sum_i (-1)^i \dim H^i(G, \boldsymbol{F}_p) \neq 0.$$

Then each open subgroup of G has trivial centralizer in G. In particular, the center of G is trivial.

Observing that the conditions (1)–(3) are inherited by any open subgroup of G, we see that we may prove just the center-triviality of G. The proof is divided into two steps.

Step 1. Let $\Lambda = \boldsymbol{Z}_p[[G]]$ be the complete group algebra of G over the ring of p-adic integers $\boldsymbol{Z}_p$. Then Λ is a local pseudocompact ring whose unique open maximal ideal R is the kernel of the canonical augmentation $\Lambda \to \boldsymbol{Z}/p\boldsymbol{Z}$. The following 'Nakayama lemma' due to A. Brumer [1] plays a crucial role in this step.

Lemma 2 (Brumer). *Let Λ be a pseudocompact ring with radical R, M a pseudocompact Λ-module, and let $x_1, \ldots, x_m \in M$. If M/RM is* (*topologically*) *generated by the images of $x_1, \ldots, x_m$, then $M = \Lambda x_1 + \cdots + \Lambda x_m$.*

Proof. See [1] Corollary 1.5.

It is remarkable that, in contrast to the usual Nakayama lemma, the above Brumer's lemma does not assume the finite generation of M as a Λ-module, but does imply it.

Lemma 3. *Let G be a pro-p group satisfying the conditions* (1),(2) *of Theorem* 1. *Then the trivial Λ-module $\boldsymbol{Z}_p$ has a finite free resolution:*

$$(F): \quad 0 \to F_n \to F_{n-1} \to \cdots \to F_0 \to \boldsymbol{Z}_p \to 0,$$

where each F_i is a free Λ-module of finite rank $(0 \leq i \leq n)$.

Proof. We shall follow an argument in Gruenberg [3] 8.1 carefully in our context.

1°. We first show by induction on $N \geq 1$ that there is an exact sequence of Λ-modules

$$(A_N): \quad 0 \to K_N \to F_{N-1} \to \cdots \to F_0 \to \boldsymbol{Z}_p \to 0,$$

in which F_i $(0 \leq i \leq n-1)$ are free of finite ranks and K_N is arbitrary. If $N = 1$, then we can take as $F_0 = \Lambda$, $K_1 =$ the augmentation ideal of Λ. So we assume that the exact sequence (A_N) is obtained. To obtain (A_{N+1}), it suffices to show that K_N in the sequence (A_N) is finitely generated. As the