66. *On the Uniform Distribution Modulo One of Some Log-like Sequences*

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1. Introduction and main results. Let p_n denote the nth prime number. Let f be a polynomial with real coefficients, then it is known that the sequence $\{f(p_n)\}_{n=1}^{\infty}$ is uniformly distributed modulo one (u.d. mod 1) if and only if f is an irrational polynomial, which means that the polynomial $f(x) - f(0)$ has one irrational coefficient at least. (cf. [3]). Furthermore, it is also known that for any noninteger $\alpha \in (0, \infty)$, the sequence $\{p_n^{\alpha}\}_{n=1}^{\infty}$ is u.d. mod 1 (see e.g. [1], [6]).

On the other hand, Goto and Kano [2] investigated the log-like functions f and obtained sufficient conditions on the function f for which the sequence $\{f(p_n)\}_{n=1}^{\infty}$ is u.d. mod 1. Unfortunately we could not underestand the proof of main Theorem 2. In this paper we first modify Goto and Kano's results (see Theorems 1 and 2 below) and then give a new result (Theorem 3). The proofs are given in Section 2. (Though our Theorem 1 is essentially the same as Theorem 1 of [2], we give here a proof for completeness' sake.)

Theorem 1. Let $a > 0$ and let $f : [a, \infty) \to (0, \infty)$ be a differentiable function. Assume that $xf'(x) \to \infty$ as $x \to \infty$ and that for sufficiently large x, $(\log x) f'(x)$ is monotone in x. Further, assume that for some $\varepsilon > 0$, $f(x) = o((\log x)^{\varepsilon})$ as $x \to \infty$. Then the sequence $\{\alpha f(p_n)\}_{n=n_0}^{\infty}$ is u.d. mod 1, where $n_0 = min\{n : p_n > a\}$ and α is any nonzero real constant.

Theorem 2. Let $a > 0$ and let $f : [a, \infty) \to (0, \infty)$ be a twice differentiable function with $f' > 0$. Assume that $x^2 f''(x) \to \infty$ as $x \to \infty$ and that for sufficiently large x, $(\log x)^2 f''(x)$ is nonincreasing in x. Further, assume that for some $\varepsilon > 0, f(x) = o((\log x)^{\varepsilon})$ as $x \to \infty$. Then the sequence $\{\alpha f(p_n)\}_{n=n_0}^{\infty}$ is u.d. mod 1, where $n_0 = min\{n : p_n > a\}$ and α is any nonzero real constant.

Theorem 3. Let $a > 0$ and let $f : [a, \infty) \to (0, \infty)$ be a twice differentiable function with $f' > 0$. Assume that $x^2 f''(x) \to -\infty$ as $x \to \infty$ and that for sufficiently large x, both $(\log x)^2 f''(x)$ and $x(\log x)^2 f''(x)$ are nondecreasing in x. Further, assume that for some $\varepsilon > 0, f(x) = o((\log x)^{\varepsilon})$ as $x \to \infty$. Then the sequence $\{\alpha f(p_n)\}_{n=n_0}^{\infty}$ is u.d. mod 1, where $n_0 = min\{n : p_n > a\}$ and α is any nonzero real constant.

Note that Theorem 2 is essentially concerned with a convex function f, while Theorem 3 is concerned with a concave function f. Applying Theorem 3 to the function $f(x) = (\log x)^{\varepsilon}$ we obtain that the sequence $\{(\log p_n)^{\varepsilon}\}_{n=1}^{\infty}$ is u.d. mod 1 if $\varepsilon > 1$.

2. The proofs. We first prove Theorem 3 and then prove Theorems 1