51. *Description of Sequences Defined by Billiards in the Cube*

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§ 1. **Introduction.** We consider billiards in the cube I^3, $I=[0,1]$, whose faces $\{\delta\}\times I\times I$, $I\times\{\delta\}\times I$, and $I\times I\times\{\delta\}$ are labelled by a, b, and c, resp., where $\delta=0$ or 1 and $A\times B\times C=\{(x,y,z)\mid x\in A, y\in B, z\in C\}$. Let a particle start at a point $P\in F$ with constant velocity along a vector $v=(1,\alpha,\beta)$ and reflected at each face specularly, where $F=\{0\}\times I'\times I'\cup I'\times\{0\}\times I'\cup I'\times I'\times\{0\}$, $I'=[0,1)$. We assume that
(A) α, β, β/α are irrational with $1>\alpha>\beta>0$, and
(B) the (forward) path of the particle never touch the edges of the cube.
A point $P\in F$ of the property (B) will be called *lattice-free* w.r.t, a given v. If we write down the labels a, b, and c of the faces which the particle hits in order of collision, we have an infinite sequence, or word,

$$w=w(v,P)=w(v,P;a,b,c)\in\{a,b,c\}^{N}.$$

In [1] the authors jointly with P. Arnoux and C. Mauduit proved the following theorem conjectured by G. Ranzy: If 1, α, β are linearly independent over Q and if $P\in F$ is lattice-free w. r. t. v, then the *complexity* $p(n;w)$ of the word $w=w(v,P)$ is given by

$$p(n;w)=n^2+n+1 \quad (n\geq 1),$$

where $p(n;w)$ is, by definition, the number of distinct subwords of w of length n. The purpose of this note is to give an algorithm describing the word w in terms of the *partial quotients* of the simple continued fractions of α, β, and β/α and the *digits* appearing in certain expansions, defined by (4) below, of the coordinates of the point P.

By symmetry with respect to the faces, the word w remains unchanged, if we replace the cube by the three dimensional torus $\boldsymbol{R}^3/\boldsymbol{Z}^3$ and imagine that the particle does not reflect at the faces but passes through them. If we attach the symbols a, b, and c to the intersection points of the half-line $l=\{tv+P\mid t>0\}$ to the planes $x=k\in\boldsymbol{N}$, $y=m\in\boldsymbol{N}$, and $z=n\in\boldsymbol{N}$, resp., and trace them along l, we obtain the word $w(v,P)$ defined above. We remark that a point $P=(\xi,\eta,\zeta)\in F$ is lattice-free w.r.t. $v=(1,\alpha,\beta)$ if and only if

(1) $$k\theta_i+\phi_i\notin\boldsymbol{Z} \text{ for all } k\in\boldsymbol{N}\ (i=1,2,3),$$

where

(2) $$\theta_1=\alpha,\ \phi_1=\eta-\alpha\xi,\ \theta_2=\beta,\ \phi_2=\xi-\beta\xi,\ \theta_3=\frac{\beta}{\alpha},\ \phi_3=\xi-\frac{\beta}{\alpha}\eta,$$

and that almost all points $P\in F$ in the sense of Lebesgue Measure are lattice-free w.r.t. a given vector v.

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