49. On a Problem of Dinaburg and Sinai

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§1. **Introduction.** Let N be a sufficiently large integer. Let

$$F_N = \{a/b\,;\,1 \le a \le b \le N,\ (a,\,b) = 1,\ a \text{ and } b \text{ are integers}\}.$$

For any fraction a/b in F_N, we can associate the minimum positive integer $x_0 \le b$ such that

$$|\,ax_0 - by_0\,| = 1$$

for some integer $y_0 \ge 1$. Let α_1, β_1, α_2 and β_2 be real numbers satisfying

$$0 < \alpha_1 < \beta_1 < \alpha_2 < \beta_2 < 1.$$

Then we put

$$S_N = \{a/b \in F_N\,;\,\alpha_1 N < a < \beta_1 N < \alpha_2 N < b < \beta_2 N\}.$$

Dinaburg and Sinai [1] have studied the distribution of

$$x_0/b$$

as a/b belongs to S_N and $N \to \infty$. We shall improve both their results and Remark by Voronin and Tvnek in p.171 of [1].

For any a/b in F_N, we may associate the minimum positive integer $x_1 \le b$ such that

$$ax_1 - by_1 = 1$$

for some integer $y_1 \ge 1$. We may also treat the distribution of

$$x_1/b$$

as a/b belongs to F_N or S_N and $N \to \infty$.

We may describe x_0/b in two ways. For $(a,\,b) = 1$, let $\bar{a}$ be the unique positive integer $\le b$ such that $a\bar{a} \equiv 1(mod\ b)$. By the definition of x_0, we see first that

$$x_0 = Min(\bar{a},\,b - \bar{a}).$$

Wenext express x_0/b in terms of the continued fraction expansion of a/b. We denote

$$\cfrac{1}{a_1 + \cfrac{1}{a_2 + \ddots + \cfrac{1}{a_n}}}$$

by $[a_1,\,a_2,\ldots,a_n]$ and also by p_n/q_n for $n \ge 1$, where $a_1,\,a_2,\ldots$ and a_n are positive integers. We define $p_0 = 0$ and $q_0 = 1$. Now suppose that

$$a/b = [a_1,\,a_2,\ldots,\,a_s]$$

with the minimum integer $s \ge 1$. Thus we suppose that $a_s \ge 2$ unless $a/b = 1$. When s is odd, then $p_s q_{s-1} - q_s p_{s-1} = (-1)^{s+1}$ with $p_s = a$, $q_s = b$ and $q_{s-1} = \bar{a}$. Thus

$$x_0/b = \bar{a}/b = q_{s-1}/q_s = [a_s,\,a_{s-1},\ldots,\,a_2,\,a_1].$$

When s is even, then $p_s = a$, $q_s = b$ and $q_{s-1} = b - \bar{a}$. Thus in this case we