43. *Some Remarks on the Fifth Painlevé Equation on the Positive Real Axis*

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(Communicated by Kunihiko KODAIRA, M. J. A., Sept. 14, 1992)

Consider an equation of the form

$$\text{(V)}\quad y''=\left(\frac{1}{2y}+\frac{1}{y-1}\right)y'^2-\frac{y'}{x}+\frac{\alpha}{x^2}(y-1)^2y+\frac{\gamma y}{x}-\frac{\delta y(y+1)}{y-1}$$

($'=d/dx$) on the positive real axis $x>0$, where α, γ and δ are real constants. This is a special case of the fifth Painlevé equation. If $\delta>0$, equation (V) admits a one-parameter family of solutions $\{Y(a,x);\ a\in \boldsymbol{R}\}$ satisfying $Y(a,x)\simeq ae^{-\sqrt{2\delta}\,x}x^{-\gamma/\sqrt{2\delta}-1}$ as $x\to+\infty$. Furthermore any real-valued solution $\varphi(x)$ satisfying $\varphi(x)\to 0$ as $x\to+\infty$ is written in the form $\varphi(x)=Y(a_0,x)$, where a_0 is some real constant (cf. [1]). In this note we show the existence of families of solutions with analogous properties near the regular singular point $x=0$.

1. We treat the following equations equivalent to (V).

Proposition ([1; Proposition 2.2]). *By* $y=\tanh^2 u$, *equation* (V) *is changed into*

$$\text{(E.0)}\qquad x(xu')'=\frac{\alpha}{2}\tanh u\cosh^{-2}u+\frac{\gamma}{4}x\sinh 2u+\frac{\delta}{8}x^2\sinh 4u$$

and, by $y=-\tan^2 u$, *equation* (V) *is changed into*

$$\text{(E.}-\text{)}\qquad x(xu')'=\frac{\alpha}{2}\tan u\cos^{-2}u+\frac{\gamma}{4}x\sin 2u+\frac{\delta}{8}x^2\sin 4u.$$

We obtain a one-parameter family of solutions near $x=0$.

Theorem 1. *Assume that* $\alpha>0$. *Then, for an arbitrary positive constant* C_0, *equation* (V) *admits a family of real-valued solutions* $\{Y_0(c,x);\ -C_0<c<C_0\}$ *satisfying*

$$Y_0(c,x)=cx^{\sqrt{2\alpha}}(1+O(x+|c|x^{\sqrt{2\alpha}})),$$
$$(d/dx)\,Y_0(c,x)=\sqrt{2\alpha}\,cx^{\sqrt{2\alpha}-1}(1+O(x+|c|x^{\sqrt{2\alpha}}))$$

on the interval $0<x<r_0$, *where* $r_0=r_0(C_0)$ *is a sufficiently small positive constant.*

Proof. Equation (E.0) is written in the form

$$(1)\qquad x(xu')'=u\left(\frac{\alpha}{2}+F_0(x,u)\right),$$

where $F_0(x,u)=O(x+u^2)$ for $|u|<1$, $0<x<1$. By $u=x^{\sqrt{\alpha/2}}w$ equation (1) is changed into

$$(2)\qquad x(xw')'+\sqrt{2\alpha}\,xw'=wF(x,w),$$

where

$$(3)\qquad F(x,w)=O(x+x^{\sqrt{2\alpha}}w^2)$$

for $|w|<x^{-\sqrt{\alpha/2}}$, $0<x<1$. Consider a system of integral equations of the