32. Some Problems of Diophantine Approximation in the Theory of the Riemann Zeta Function

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§ Introduction. Let \( \alpha \) be a positive number. The distribution of the fractional part \( \{\alpha n\} \) of \( \alpha n \) has been studied extensively. It is well-known that it depends heavily on the arithmetic nature of \( \alpha \). We may briefly recall this fact for a quadratic irrational \( \alpha \) as follows. It was shown by Hardy-Littlewood [6] and Ostrowski [8] that

\[
\sum_{n \leq X} \left( \{\alpha n\} - \frac{1}{2} \right) \ll \log X.
\]

Hecke [7] has shown, in fact, that if \( \alpha \) is \( \sqrt{D} \) or \( 1/\sqrt{D} \) with a positive square free integer \( D = 2 \) or \( 3 \) (mod 4), then for any \( \varepsilon > 0 \)

\[
\sum_{n \leq X} \left( \{\alpha n\} - \frac{1}{2} \right) \log^a \frac{X}{n} = A_1 \log^a X + A_2 \log^a X + A_3 \log X
\]

\[+ \sum_{m = 1}^{\infty} C_m X^{(\log \eta_m)/(\log \sqrt{D})} + O(X^{-1 + \varepsilon}),
\]

where \( A_1, A_2, A_3 \) and \( C_m \) are some constants, \( C_m = O(|m|^{-2 + \varepsilon}) \) for \( m \neq 0 \) and \( \eta_m \) is the fundamental unit of the quadratic number field \( Q(\sqrt{D}) \) or the square of it. The author [4] [5] has extended his result and shown that for any \( \varepsilon > 0 \)

\[
\sum_{n \leq X} \left( \{\alpha n\} - \frac{1}{2} \right) \log \frac{X}{n} = \frac{1}{2} G_1(\alpha) \log^a X + G_2(\alpha) \log X
\]

\[+ \sum_{m = 1}^{\infty} C'_m X^{(\log \eta_m)/(\log \sqrt{D})} + O(X^{-1/2 + \varepsilon}),
\]

where \( G_1(\alpha) \) and \( G_2(\alpha) \) can be explicitly written down in terms of the continued fraction expansion of \( \alpha \) and \( C'_m = O(|m|^{-1/2 + \varepsilon}) \) for \( m \neq 0 \).

Here we are concerned with the distribution of

\[
\left\{ \frac{\alpha - \gamma}{2\pi} \right\} - \frac{1}{2},
\]

where \( \gamma \) runs over the positive imaginary parts of the zeros of the Riemann zeta function \( \zeta(s) \). Our main problem is to find an asymptotic formula for the sum

\[
\sum_{\gamma \leq \frac{X}{2\pi}} \left( \left\{ \frac{\alpha - \gamma}{2\pi} \right\} - \frac{1}{2} \right)
\]

and determine how it depends on \( \alpha \). Our result is not precise enough for

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