25. On Certain Real Quadratic Fields with Class Number 2

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Let D be a square-free rational integer and $\varepsilon_D=(t+u\sqrt{D})/2$ $(t, u>0)$ be the fundamental unit of $\boldsymbol{Q}(\sqrt{D})$ with $N\varepsilon_D=-1$, where N is the norm map from $\boldsymbol{Q}(\sqrt{D})$ to $\boldsymbol{Q}$. Then D is expressed in the form $D=u^2n^2\pm 2an+b$, where n, a and b are integers such that $n\geqq 0$, $0\leqq a<u^2/2$ and $a^2+4=bu^2$ (cf. [6]). We denote by $h(D)$ the class number of $\boldsymbol{Q}(\sqrt{D})$. In our previous paper [1], we treated the problem of enumerating the real quadratic fields $\boldsymbol{Q}(\sqrt{D})$ with $h(D)=1$ and $1\leqq u\leqq 300$ (the cases $u=1$ and $u=2$ were treated in [3]).

In this paper, we shall consider the same problem for real quadratic fields $\boldsymbol{Q}(\sqrt{D})$ with $h(D)=2$ and $1\leqq u\leqq 200$.

We note here that the list in [4] is incomplete as it misses $\boldsymbol{Q}(\sqrt{3365})$ whereas $h(3365)=2$.

In the same way as in [1], we have the following theorem.

Theorem. *With the notation as above, there exist* 45 *real quadratic fields* $\boldsymbol{Q}(\sqrt{D})$ *with class number two for* $1\leqq u\leqq 200$, *where* D *are those in table with one possible exception.*

Proof. Let d be the discriminant of $\boldsymbol{Q}(\sqrt{D})$, that is, $d=D$ or $4D$, according as $D\equiv 1 \pmod 4$ or not. Let χ_d be the Kronecker character belonging to $\boldsymbol{Q}(\sqrt{D})$ with the discriminant d and $L(s,\chi_d)$ be the corresponding L-series. Then by Theorem 2 of [5], we have for any $y\geqq 11.2$ satisfying $e^y\leqq d$

$$L(1,\chi_d)>\frac{0.655}{y}d^{-1/y}$$

with one possible exception of d.

Hence from class-number formula, we have

$$h(D)=\frac{\sqrt{d}}{2\log\varepsilon_D}L(1,\chi_d)>\frac{0.655}{y}\,\frac{\sqrt{d}\,d^{-1/y}}{2\log(u\sqrt{d})}$$
$$\geqq\frac{0.655\,e^{(y/2)-1}}{y(y+2\log u)}.$$

Put for convenience

$$g(\log u, y)=\frac{0.655\,e^{(y/2)-1}}{y(y+2\log u)}.$$

Then $g(\log u, y)$ is a monotone increasing function for $y\geqq 11.2$. Therefore for any fixed u, there exists a real number $c=c(u)$ such that $c\geqq 11.2$