18. *Affirmative Solution of a Conjecture Related to a Sequence of Shanks*

By R. A. MOLLIN*) and H. C. WILLIAMS**)

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Abstract: In [6] the authors conjectured that if $d\equiv 1 \pmod 8$ is positive, square-free and all Q_i's (see below) are powers of 2 in the continued fraction expansion of $(1+\sqrt{d})/2$ then the class number $h(d)$ of $Q(\sqrt{d})$ is equal to 1 if and only if $d\in\{17, 41, 113, 353, 1217\}$. The purpose of this note is to prove this conjecture and show how it relates to results in the literature including work of Shanks [7] concerning certain special forms. Moreover we solve the class number 2, 3, and 4 problems for these forms. Finally, we leave a conjecture for other forms at the end.

§ 1. **Notations and preliminaries.** Let d be a positive square-free integer and let $w_d=(\sigma-1+\sqrt{d})/\sigma$ where $\sigma=\begin{cases}1 \text{ if } d\equiv 2, 3 \pmod 4\\ 2 \text{ if } d\equiv 1 \pmod 4\end{cases}$. The discriminant of $K=Q(\sqrt{d})$ is $\Delta=(2/\sigma)^2 d$, and the maximal order in K is denoted $\mathcal{O}_K$. Let $w_d=\langle a, \overline{a_1, a_2, \cdots, a_k}\rangle$ be the continued fraction expansion of w_d. Here $a_0=a=\lfloor w_d\rfloor$, (where $\lfloor x\rfloor$ denotes the greatest integer less than or equal to x); and $a_i=\lfloor (P_i+\sqrt{d})/Q_i\rfloor$ for $i\geq 1$ where: $(P_0, Q_0)=(\sigma, \sigma-1)$ and $P_{i+1}=a_iQ_i-P_i$; $Q_{i+1}Q_i=d-P_{i+1}^2$ for $i\geq 0$.

The Legendre symbol will be denoted by $(/)$. Finally for the theory of reduced ideals used herein the reader is referred to [5] or [8].

§ 2. **Q_i's as powers of 2.** The conjecture posed in [6] is that any square-free $d\equiv 1 \pmod 8$ with all Q_i's as powers of 2 and $h(d)=1$ can only hold for $d\in\{17, 41, 113, 252, 1217\}$. In [1] we classified for a general square-free d all those forms for which all the Q_i/Q_0's are powers of a given integer $c>1$. In particular for the case where $d\equiv 1 \pmod 8$ and all the $\mathcal{O}_K$-primes above 2 are principal then all Q_i's are powers of 2 if and only if $d=(2^s+1)^2+2^{s+2}$, where $s>0$ and $k=1+2s$.

Theorem 2.1. *If $d\equiv 1 \pmod 8$ and all Q_i's are powers of 2 then $h(d)=1$ if and only if $d\in\{17, 41, 113, 353, 1217\}$.*

Proof. We will now show the remarkable fact that d is a quadratic residue of 127 if $d=(2^n+1)^2+2^{n+2}$, (observe: $127=2^7-1$).

Let $n\equiv m_0 \pmod 7$ where $0\leq m_0\leq 6$.

If $m_0=0$ then $d\equiv 32^2 \pmod{127}$;

If $m_0=1$ then $d\equiv 25^2 \pmod{127}$;

*) Department of Mathematics and Statistics, University of Calgary, Calgary, Alberta, Canada, T2N IN4.

**) Department of Computer Science, University of Manitoba, Winnipeg, Manitoba, Canada, R3T 2N2.