13. *Contributions to Uniformly Distributed Functions. I*

*Discrepancy of Fractal Sets**),**)

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(Communicated by Shokichi IYANAGA, M. J. A., Feb. 12, 1991)

1. **Introduction.** Let (M, d) denote a compact, arcwise connected metric space and μ a positive, regular, normalized Borel measure on M. A continuous function x: $[0, \infty)\to M$ is called *μ-uniformly distributed* (for short: μ-u.d.) if

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T f(x(t))dt=\int_M f(x)d\mu(x) \tag{1.1}$$

holds for all continuous, real-valued functions f on M. In the case of the K-dimensional torus $M=(R/Z)^K$ (with Lebesgue measure μ) E. Hlawka [7] introduced a quantitative measure for u.d., the so called *discrepancy*

$$D_T(x) := \sup_J \left|\frac{1}{T}\int_0^T \chi_J(x(t))dt-\mu(J)\right|, \tag{1.2}$$

where the supremum is extended over all intervals J parallel to the coordinate axes and χ_J denotes the characteristic function of J. It is an easy observation that $x(t)$ is u.d. if and only if $D_T(x)$ tends to 0 for $T\to\infty$. For a detailed survey on the theory of uniform distribution we refer to the monographs [10] and [8]. In [4] we have generalized the concept of discrepancy to compact metric spaces and obtained some lower bounds. Instead of the family of intervals (parallel to the coordinate axes) more general classes of subsets were considered: so-called *discrepancy systems.*

In the context of the present paper a discrepancy system $\mathcal{D}$ is a family of measurable subsets $E\subseteq M$ satisfying the following condition: *For every open ball $B(x, r)$ with center $x\in M$ and radius $r>0$ there exist a set $E\in\mathcal{D}$ and a ball $B(x, R)$ such that*

$$B(x, r)\subseteq E\subseteq B(x, R) \quad \textit{and} \quad R/r\leq\beta \tag{1.3}$$

with an absolute constant β.

Furthermore we assume the following additional property for the measure μ: *There exists a constant $K>1$ such that*

$$\mu(B(x, r))\leq\alpha r^K \tag{1.4}$$

for every open ball $B(x, r)$, α denoting an absolute constant.

Then for every continuous function x: $[0, \infty)\to M$, the discrepancy (with respect to $\mathcal{D}$) can be defined by

$$D_T(\mathcal{D}, x) := \sup_{E\in\mathcal{D}} \left|\frac{1}{T}\int_0^1 \chi_E(x(t))dt-\mu(E)\right|. \tag{1.5}$$

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*) Dedicated to Prof. E. Hlawka on the occasion of his 75th birthday.

**) This paper was supported by the Austrian Science Foundation project P 8274.