9. *One Criterion for Multivalent Functions*

By Mamoru NUNOKAWA and Shinichi HOSHINO
Department of Mathematics, University of Gunma

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Let P be the class of functions $p(z)$ which are analytic in the unit disk $E=\{z: |z|<1\}$, with $p(0)=1$ and $\operatorname{Re} p(z)>0$ in E.

If $p(z)\in P$, we say $p(z)$ is a *Carathéodory function*. It is well-known that if $f(z)=z+\sum_{n=2}^{\infty}a_n z^n$ is analytic in E and $f'(z)\in P$, then $f(z)$ is univalent in E [1, 8].

Ozaki [5, Theorem 2] extended the above result to the following:

If $f(z)$ is analytic in a convex domain D and

$$\operatorname{Re}(e^{i\alpha}f^{(p)}(z))>0 \qquad \text{in } D$$

where α is a real constant, then $f(z)$ is at most p-valent in D.

This shows that if $f(z)=z^p+\sum_{n=p+1}^{\infty}a_n z^n$ is analytic in E and

$$\operatorname{Re} f^{(p)}(z)>0 \qquad \text{in } E,$$

then $f(z)$ is p-valent in E.

Nunokawa improved the above result to the following:

Theorem A. *Let $p\geqq 2$. If $f(z)=z^p+\sum_{n=p+1}^{\infty}a_n z^n$ is analytic in E and*

$$|\arg f^{(p)}(z)|<\frac{3}{4}\pi \qquad \text{in } E,$$

then $f(z)$ is p-valent in E (cf. [3]).

Theorem B. *Let $p\geqq 2$. If $f(z)=z^p+\sum_{n=p+1}^{\infty}a_n z^n$ is analytic in E and*

$$\operatorname{Re} f^{(p)}(z)>-\frac{\log(4/e)}{2\log(e/2)}p! \qquad \text{in } E,$$

then $f(z)$ is p-valent in E (cf. [4]).

In this paper, we need the following lemmas.

Lemma 1 ([6], Lemma 4). *Let $p(z)$ be analytic in E with $p(0)=1$ and $\operatorname{Re} p(z)>1/2$ in E.*

Then for any function $f(z)$, analytic in E, the function $p(z)*f(z)$ takes its values in the convex hull of $f(z)$, where $p(z)*f(z)$ denotes the convolution or Hadamard product of $p(z)$ with $f(z)$.

Lemma 2 ([7]). *Let $p(z)$ be analytic in E with $p(0)=1$. Suppose that $\alpha>0$, $\beta<1$ and that for $z\in E$, $\operatorname{Re}(p(z)+\alpha zp'(z))>\beta$.*

Then for $z\in E$,

$$\operatorname{Re} p(z)>1+2(1-\beta)\sum_{n=1}^{\infty}\frac{(-1)^n}{1+\alpha n}.$$

The estimate is best possible for

$$p_0(z)=2\beta-1+2(1-\beta)\sum_{n=1}^{\infty}\frac{z^n(-1)^n}{1+\alpha n}.$$

Proof. For $z\in E$, write $p(z)=1+\sum_{n=1}^{\infty}p_n z^n$, so that