8. *Formation of Singularities in Solutions of the Nonlinear Schrödinger Equation**)

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§ 1. **Introduction and results.** This paper is a sequel to the previous ones [5] and [6]. We continue the study of the L^2-concentration in solutions of initial value problem for the nonlinear Schrödinger equation:

$$(\mathrm{Cp})\quad \begin{cases} (\mathrm{NLS}) & 2i\dfrac{\partial u}{\partial t}+\Delta u+|u|^{4/N}u=0, \qquad (t,x)\in \boldsymbol{R}^+\times \boldsymbol{R}^N, \\ (\mathrm{IV}) & u(0,x)=u_0(x), \qquad x\in \boldsymbol{R}^N, \end{cases}$$

where $i=\sqrt{-1}$, $u_0\in H^1=H^1(\boldsymbol{R}^N)$, Δ is the Laplacian on $\boldsymbol{R}^N$.

The local existence theory for (Cp) is well known ([1], [3]); there are $T_m\in(0,\infty]$ (maximal existence time) and a unique solution $u(\cdot)\in C([0,T_m);H^1)$ of (Cp). Furthermore u satisfies

$$(1.1)\qquad \|u(t)\|=\|u_0\|,$$

$$(1.2)\qquad E(u(t))\equiv\|\nabla u(t)\|^2-(2/\sigma)\|u(t)\|_\sigma^\sigma=E(u_0),$$

for $t\in[0,T_m)$. Here $\sigma=2+4/N$ and $\|\cdot\|$ ($\|\cdot\|_\sigma$) denotes the $L^2(\boldsymbol{R}^N)(L^\sigma(\boldsymbol{R}^N))$-norm.

It is also well-known (see [2]) that, for some u_0, the solution u shows the singular behavior (blow-up) that

$$(1.3)\qquad \lim_{t\to T_m}\|\nabla u(t)\|=\|u(t)\|_\sigma=\infty$$

for some $T_m\in(0,\infty]$.

Of physical importance is the case $N=2$, when (NLS) is a model of the stationary self-focusing of a laser beam propagating along the t-axis. It is considered that the singular behavior (1.3) corresponds to the focus of the beam. Thus our purpose is to obtain more precise analysis of the behavior of the singular solution $u(t)$ of (Cp) as $t\uparrow T_m$. Because of its mathematical interest however, we intend to develop a theory for arbitrary dimensions N. It should be noted that (NLS) has a remarkable property that it is invariant under the pseudo-conformal transformations.

In [6], we proved;

Proposition A. *Suppose that the solution $u(t)$ of (Cp) satisfies (1.3). Let $(t_n)_n$ be any sequence such that $t_n\to T_m$ as $n\to\infty$. Set*

$$(\mathrm{A.1})\qquad \lambda_n\equiv\lambda(t_n)=1/\|u(t_n)\|_\sigma^{\sigma/2}\qquad(\longrightarrow 0 \text{ as } n\longrightarrow\infty),$$

$$(\mathrm{A.2})\qquad u_n(t,x)\equiv S_{\lambda_n}u(t,x)=\lambda_n^{N/2}u(t,\lambda_n x).$$

Then there exists a subsequence of $(t_n)_n$ (we still denote it by $(t_n)_n$) which satisfies the following properties: one can find $L\in\boldsymbol{N}\cup\{\infty\}$ and sequences $(y_n^j)_n$ in $\boldsymbol{R}^N$ for $1\leqq j\leqq L$ such that

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*) In memory of my father.