60. A Proof of the Theorem of Eakin-Nagata

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(Communicated by Shokichi IYANAGA, M. J. A., Sept. 12, 1991)

In this article, we mean by a *ring* a commutative ring with identity. We are giving a new proof of the following theorem, which is known as the theorem of Eakin-Nagata ([2], [3]).

Theorem. *Let A be a subring of a noetherian ring R. If R is a finite A-module, then A is noetherian.*

To begin with, we recall the following theorem of Cohen [1].

Theorem of Cohen. *A ring A is noetherian if (and only if) every prime ideal of A has a finite basis.*

Proof. Assume the contrary, and let F be the set of ideals of A which have no finite bases. By Zorn's lemma, F has a maximal member, say, I. By our assumption, I is not a prime ideal, and there are elements b, c of A which are not in I and such that $bc \in I$. By the maximality of I, ideals $I:b$, $I+bA$ have finite bases, say, $e_1, \cdots, e_m$ and $b, f_1, \cdots, f_n$ with $f_i \in I$. Then I is generated by $f_1, \cdots, f_n, be_1, \cdots, be_m$, a contradiction.

Now we prove the theorem of Eakin-Nagata. As is easily seen, it suffices to prove it under

(additional assumption 1) $R=A[b]$ with $b \in R$.

By induction on the largeness of ideals of R, we may assume

(additional assumption 2) If J is a non-zero ideal of R, then $A/(J \cap A)$ is noetherian.

(1) Assume first that A is an integral domain. Then, we may assume that R is an integral domain. Then, there is an element c of A such that (i) $A[cb]$ is a free A-module and (ii) the field of fractions of $A[cb]$ coincides with that of R. If we see that $A[cb]$ is noetherian, then we see easily that A is noetherian, because $A[cb]$ is a free A-module. Thus we may assume that the field fractions of A coinsides with that of R. Then, there is a non-zero element d of A such that $dR \subseteq A$. By the theorem of Cohen, we have only to show that an arbitrary prime ideal P ($\neq \{0\}$) of A has a finite basis. We may choose d from elements of P. A/dR is noetherian by our assumption, and P modulo dR has a finite basis. Since R is a finite A-module, so is dR, too. Thus P has a finite basis, and A is noetherian.

(2) Assume now that A contains a zero-divisor c ($\neq 0$). Let P be an arbitrary prime ideal of A. We may choose c from elements of P. Consider cR as an A-module. This is really $A/(0:cA)$-module, and $0:cA=(0:cR) \cap A$. Therefore cR is a noetherian module by our assumption, and its submodule $cR \cap A$ has a finite basis. Since $A/(cR \cap A)$ is noetherian by our assumption, we see that P has a finite basis. Q.E.D