56. On Smooth Quartic Embedding of Kummer Surfaces

By Isao NARUKI
Research Institute for Mathematical Sciences, Kyoto University

(Communicated by Kunihiko KODAIRA, M. J. A., Sept. 12, 1991)

1. The purpose of the note is to show the following fact: *For any abelian surface admitting a polarization with the reduced pfaffian three, one can always construct a birational morphism of the associated Kummer surface into* $P_3(C)$ *whose image is a quartic surface. The morphism is a smooth embedding if the abelian surface can not be principally polarized.*

We will also discuss some geometry around this fact. Let A be an abelian surface, E the universal cover of A and G the lattice such $A=E/G$. Suppose that a polarization (ample line bundle) H is given to A. We identify H with its Riemann form. (See Weil [4].) H is thus a hermitian form on E whose imaginary part is Z-valued over G. We assume that the reduced pfaffian of H is three, that is, that the determinant of the imaginary part over G is equal to nine. We denote the space of *odd* theta functions of type $(2H, \mathbf{1})$ ($\mathbf{1}$: the trivial semi-character) by V. V is then four-dimensional. Since the semi-character is trivial, the theta functions in V necessarily vanish at the two-torsion points of A. Regarding V as a subspace of $\Gamma(A, H)$ we obtain a rational mapping of A into $\boldsymbol{P}(V^*)\simeq \boldsymbol{P}_3(\boldsymbol{C})$. ($V^*$ is the dual space of V. For a complex vector space W we denote by $\boldsymbol{P}(W)$ the projective space $(W\backslash\{0\})/\boldsymbol{C}^{\times}$.) Let S be the Kummer surface associated with A i.e. the minimal desingularization of the quotient of A by the involution $z\leftrightarrow -z$. Since all elements in V are *odd*, the rational mapping induces a rational mapping of S into $\boldsymbol{P}(V^*)$ and we see that this mapping is actually a birational morphism and that the image is a quartic surface. The line bundle $2H$ induces a line bundle over S, which we denote by $(2H)$. This is of self intersection twelve and is orthogonal to the exceptional divisors E_i $(i=1, 2, \cdots, 16)$ of the desingularization S. The line bundle

$$L:=(2H)-\frac{1}{2}\sum_{i=1}^{16} E_i$$

gives the quartic polarization of S. Since $(L, E_i)=1$, the image of E_i is a line, which we denote by l_i. We have thus obtained the sixteen lines on the image of S which are disjoint if $S\rightarrow \boldsymbol{P}(V^*)$ is an embedding. From now on we assume that A does not have a principal polarization; we identify S with its image in $\boldsymbol{P}(V^*)$.

2. The geometry of quartic surface S is quite interesting. Besides l_i $(i=1, 2, \cdots, 16)$ there is another class of disjoint sixteen lines. They are obtained in the following way: *Let* ψ_i $(i=1, 2, \cdots, 16)$ be the semi-char-