43. Some Dolbeault Isomorphisms for Locally Trivial Fiber Spaces and Applications

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1. Let N be a paracompact complex manifold of complex dimension n, S a Stein manifold of complex dimension l and $\pi: M\to N$ a locally trivial holomorphic fiber space whose fibers are biholomorphic onto S. Put $m:=\dim_C M(=n+l)$. Let $\{D_\alpha\}$ be a locally finite open covering of N satisfying that each D_α is a coordinate open subset with the trivialization $i_\alpha: \pi^{-1}(D_\alpha)\to D_\alpha\times S$ with $\prod_\alpha^1\cdot i_\alpha=\pi$, where $\prod_\alpha^1$ denotes the projection $D_\alpha\times S\ni(a,b)\mapsto a\in D_\alpha$. Let $\{U_\sigma\}$ be a sufficiently fine and locally finite open covering of S so that each U_σ is biholomorphic onto a polydisc in C^l. We sometimes identify $\pi^{-1}(D_\alpha)$ with $D_\alpha\times S$. Let $z_\alpha=(z_\alpha^1,\cdots,z_\alpha^n)$ be a local coordinate defined on D_α and $w_\sigma=(w_\sigma^1,\cdots,w_\sigma^l)$ a local coordinate defined on U_σ. We put $\zeta_{\alpha,\sigma}^i=z_\alpha^i$ $(1\le i\le n)$ and $\zeta_{\alpha,\sigma}^{n+j}=w_\sigma^j$ $(1\le j\le l)$. Then $\zeta_{\alpha,\sigma}=(\zeta_{\alpha,\sigma}^1,\cdots,\zeta_{\alpha,\sigma}^{n+l})=(z_\alpha^1,\cdots,z_\alpha^n,w_\sigma^1,\cdots,w_\sigma^l)$ defines a local coordinate in $i_\alpha^{-1}(D_\alpha\times U_\sigma)$. For an open subset $V\subset M$, we put $\mathcal{F}(V):=\{f\mid f$ is of class C^∞ in V and for any $z\in\pi(V)$, $f\mid\pi^{-1}(z)\cap V$ is holomorphic$\}$. We denote by $\mathcal{F}$ the sheaf defined by the presheaf $\{\mathcal{F}(V)\}$. Put $V_{\alpha,\sigma}:=V\cap i_\alpha^{-1}(D_\alpha\times U_\sigma)$ and $\mathcal{F}^{r,p}(V):=\{\varphi\mid\varphi$ is a $C^\infty(r,p)$-form on V and $\varphi\mid V_{\alpha,\sigma}=\sum_{I,J}\varphi_{IJ}d\zeta_{\alpha,\sigma}^I\wedge d\bar z_\alpha^J$, $\varphi_{IJ}\in\mathcal{F}(V_{\alpha,\sigma})$ for each α and $\sigma\}$, where $I=(i_1,\cdots,i_r)$, $J=(j_1,\cdots,j_p)$, $1\le i_1<\cdots<i_r\le n+l$, and $1\le j_1<\cdots<j_p\le n$. We get the sheaf $\mathcal{F}^{r,p}$ defined by the presheaf $\{\mathcal{F}^{r,p}(V)\}$ for $0\le r\le n+l$ and $0\le p\le n$. Let Ω^r be the sheaf of germs of holomorphic r-forms on M. We have an exact sequence

$$0\to\Omega^r\to\mathcal{F}^{r,0}\to\mathcal{F}^{r,1}\to\cdots\to\mathcal{F}^{r,n}\to 0.$$

For an open subset $W\subset\pi^{-1}(D_\alpha)$, put $\mathcal{E}_\alpha^{r,p,s}(W):=\{\psi\mid\psi$ is a C^∞ $(r,p+s)$-form in W and $\varphi\mid W\cap i_\alpha^{-1}(D_\alpha\times U_\sigma)=\sum_{I,J,K}\psi_{IJK}d\zeta_{\alpha,\sigma}^I\wedge d\bar z_\alpha^J\wedge d\bar w_\sigma^K$ for each $\sigma\}$, where $K=(k_1,\cdots,k_s)$ and $1\le k_1<\cdots<k_s\le l$. The presheaf $\{\mathcal{E}_\alpha^{r,p,s}(W)\}$ makes the sheaf $\mathcal{E}_\alpha^{r,p,s}$ on $\pi^{-1}(D_\alpha)$. Then we have an exact sequence

$$0\to\mathcal{F}^{r,p}\mid\pi^{-1}(D_\alpha)\to\mathcal{E}_\alpha^{r,p,0}\to\mathcal{E}_\alpha^{r,p,1}\to\cdots\to\mathcal{E}_\alpha^{r,p,l}\to 0$$

for each α, where the mapping $\mathcal{E}_\alpha^{r,p,s}\to\mathcal{E}_\alpha^{r,p,s+1}$ is induced by the Cauchy-Riemann operator $\bar\partial_S$ on S. Solving the Cauchy-Riemann equation $\dfrac{\partial f(z,w)}{\partial\bar w_j}=g(z,w)$ with C^∞ parameter $z\in D_\alpha$ and using the standard argument for Dolbeault lemma, we can prove

$$H^q(D_\alpha\times U_\sigma,\mathcal{F}^{r,p})\cong\frac{\{\varphi\in H^0(D_\alpha\times U_\sigma,\mathcal{E}_\alpha^{r,p,q})\mid\bar\partial_S\varphi=0\}}{\bar\partial_S H^0(D_\alpha\times U_\sigma,\mathcal{E}_\alpha^{r,p,q-1})}=0$$

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