42. On the Uniform Attractivity of Solutions of Ordinary Differential Equations by Two Lyapunov Functions

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1. Introduction. Consider the ordinary differential equation

(1) $$x'=f(t,x)\ (f(t,0)=0 \text{ for all } t\in \boldsymbol{R}_+:=[0,\infty)),$$

where $f:\boldsymbol{R}_+\times\boldsymbol{R}^n\to\boldsymbol{R}^n$ is continuous.

K. Murakami and M. Yamamoto [10] have given sufficient conditions for the global attractivity and equi-attractivity of the zero solution of (1) based on Lyapunov functions with negative semidefinite derivatives. Nowadays such Lyapunov functions have been often used to investigate the asymptotic behaviour of solutions [1–16].

As is well-known, the *uniform* stability properties are of practical importance, e.g. if f satisfies a Lipschitz condition in x uniformly with respect to t, then the uniform attractivity together with uniform stability imply the total stability of the zero solution (see [12], Chapter II, Theorem 4.5).

In this paper we show that, after slightly strenthening one of them, the conditions in Murakami's and Yamamoto's theorem of the global equi-attractivity (Theorem 1 in [10]) imply also the global *uniform* attractivity. In our second theorem we can guarantee the global *equi*-attractivity under essentially weaker conditions than those of Murakami's and Yamamoto's theorem on the global attractivity (Theorem 2 in [10]).

2. Notations and definitions. We use the n-dimensional real space $\boldsymbol{R}^n$ with the Euclidean norm $|\cdot|$. If $x\in\boldsymbol{R}^n$, $F\subset\boldsymbol{R}^n$, we define the distance between x and F by $d(x,F):=\inf\{|x-y|: y\in F\}$. $B(\rho)$ and $\bar{B}(\rho)$ denote the ball of radius $\rho>0$ around the origin and its closure, respectively.

Definition 1. A measurable function $\varphi:\boldsymbol{R}_+\to\boldsymbol{R}_+$ is said to be *integrally positive* if $\int_I \varphi(s)\,ds=\infty$ for every set

(2) $$I=\bigcup_{k=1}^{\infty}[\alpha_k,\beta_k],\qquad \beta_k-\alpha_k\geq\delta>0\ (k\in N).$$

If, in addition to (2), the inequalities $\Delta\geq\beta_k-\alpha_k$ $(k\in N)$ are also required of I, then φ is called *weakly integrally positive* [3].

It is easy to see that φ is integrally positive if and only if

(3) $$\liminf_{t\to\infty}\int_t^{t+\gamma}\varphi(s)\,ds>0$$

for every $\gamma>0$. Moreover, if φ is integrally positive, then it is weakly integrally positive, but the converse is not rue (e.g. $\varphi(t):=(1+t)^{-1}$). One of the purposes of this paper is to emphasize that the weak integral positivity

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