38. *On the Sums of Digits in Integers*

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Let $r \geqq 2$ be a given integer and let

$$n=a_k a_{k-1}\cdots a_0=a_k r^k+a_{k-1}r^{k-1}+\cdots+a_0, \qquad a_h\in\{0,1,\cdots,r-1\}$$

be the r-adic expansion of a nonnegative integer n. We define the sum of digit function

$$S_r(n)=a_k+a_{k-1}+\cdots+a_0.$$

This function has been studied by many authors (cf. Stolarsky [2]).

Clements and Lindström [1] proved the following formula:

$$\sum_{m\leqq n} s_2(m)=\log_2|\det A_2(n)| \qquad (n=0,1,2,\cdots),$$

where

$$A_2(n)=(a_{ij})_{0\leqq i,j\leqq n}, \qquad a_{ij}=(-1)^{\alpha_{ij}},$$

and α_{ij} is the number of common terms $b_h=c_h=1$ in the dyadic expansions $i=\sum_{h\geqq 0} b_h 2^h$, and $j=\sum_{h\geqq 0} c_h 2^h$. In the present paper, we generalize this formula for any given base $r\geqq 2$.

We first define a matrix $A_r(n)$ for a given integer $r\geqq 2$ and any nonnegative integer n. We choose any real number ρ satisfying $1<\rho\leqq 2^{1/(r-1)}$ and define a complex number $\zeta=\zeta(\rho)$ by

$$|\zeta|=1, \qquad |\zeta-1|=\rho \quad \text{with} \quad \operatorname{Im}\zeta\geqq 0.$$

(If $r=2$, we can choose $\rho=2$ and so $\zeta=-1$.) Then we choose real numbers $\beta_h=\beta_h(\rho)$ $(h=1,2,\cdots,r-1)$ such that

$$|\zeta^{\beta_h}-1|=\rho^h \quad \text{with} \quad 1=\beta_1<\beta_2<\cdots\beta_{r-1}\leqq\frac{\pi}{\rho}.$$

Let $i=\sum_{h\geqq 0} b_h r^h$ and $j=\sum_{h\geqq 0} c_h r^h$ be the r-adic expansions of nonnegative integers i and j, and put

$$A_r(n)=(a_{ij})_{0\leqq i,j\leqq n}, \qquad a_{ij}=\zeta^{\beta_{ij}},$$

where

$$\beta_{ij}=\sum_{h\geqq 0,\, b_h=c_h\neq 0} \beta_{b_h}.$$

Theorem. *We have*

$$\sum_{m\leqq n} s_r(m)=\log_\rho|\det A_r(n)| \qquad (n=0,1,2,\cdots).$$

Proof. For any integer $n\geqq 0$ we write

(1) $$n=hr^k+m \quad (k,h,m\in Z,\ k\geqq 0,\ 0\leqq h<r,\ 0\leqq m<r^k).$$

If we put $f(n)=\sum_{m\leqq n} s_r(m)$, then we have

(2) $$f(n)=f(hr^k+m)=hf(r^k-1)+\frac{h(h-1)}{2}r^k+h(m+1)+f(m).$$

Conversely, this formula (2) defines the function $f(n)$ $(n\geqq 0)$ uniquely