23. *An Elementary Construction of Galois Quaternion Extension*

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1. Let F be a field and let $\tilde{F}$ be a (fixed) algebraic closure of F. An extension field K of F ($F \subseteq K \subseteq \tilde{F}$) will be said to be a *Galois quaternion extension* of F if K/F is a Galois extension and its Galois group $Gal(K/F)$ is isomorphic to the quaternion group of order 8.

Theorem. *Let F be a field of the characteristic $\neq 2$ and let $F(\sqrt{m})$ $(m \notin F^2 = \{x^2 \mid x \in F\})$ be a quadratic extension of F.*

Suppose,

(i) *m is a sum of 3 non-zero squares in F: $m = p^2 + q^2 + r^2$, $p, q, r \in F$, $pqr \neq 0$,*

(ii) $n = p^2 + q^2 \notin F^2$,

(iii) $mn \notin F^2$.

Let

$$\omega = \sqrt{\sqrt{mn}\,(\sqrt{m}+\sqrt{n})(\sqrt{n}+p)} \in \tilde{F}$$

where we choose $\sqrt{mn} = \sqrt{m}\sqrt{n}$.

Then $K = F(\omega)$ is a Galois quaternion extension of F.

Proof. Let $M = F(\sqrt{m}, \sqrt{n})$ be a bicyclic biquadratic extension of F and let $Gal(M/F) = \{\sigma_0 = 1_M, \sigma_1, \sigma_2, \sigma_3\}$ where $\sigma_0 = 1_M$ (the identity),

σ_1: $(\sqrt{m}, \sqrt{n}) \longrightarrow (-\sqrt{m}, \sqrt{n})$,

σ_2: $(\sqrt{m}, \sqrt{n}) \longrightarrow (\sqrt{m}, -\sqrt{n})$,

σ_3: $(\sqrt{m}, \sqrt{n}) \longrightarrow (-\sqrt{m}, -\sqrt{n})$.

Let $K = M(\omega)$ $(\omega^2 \in M)$ and let $\alpha_i: K \to \tilde{F}$ $(i=0,1,2,3)$ denote any (but fixed once for all) embeddings of K into $\tilde{F}$ which extend σ_i $(i=0,1,2,3)$ respectively.

Now, calculating

$$(\omega^{\alpha_i})^2 = (\sqrt{mn}\,(\sqrt{m}+\sqrt{n})(\sqrt{n}+p))^{\sigma_i} \qquad = 0, 1, 2, 3)$$

we have

$$\omega^{\alpha_0} = \omega e_0, \qquad \omega^{\alpha_1} = \omega \frac{\sqrt{m}-\sqrt{n}}{r} e_1,$$

$$\omega^{\alpha_2} = \omega \frac{\sqrt{m}-\sqrt{n}}{r} \frac{\sqrt{n}-p}{q} e_2, \qquad \omega^{\alpha_3} = \omega \frac{\sqrt{n}-p}{q} e_3$$

where $e_i = \pm 1$ $(i=0,1,2,3)$ are the signs depending on α_i $(i=0,1,2,3)$ respectively. Since, as seen from the above calculations, ω^{α_i} $(i=0,1,2,3)$ are all in K for any extension $\alpha_i: K \to \tilde{F}$ of σ_i $(i=0,1,2,3)$, it follows that $K = M(\omega)$ is a Galois extension of F and α_i $(i=0,1,2,3)$ are automorphisms of K