15. Invariants and Hodge Cycles. IV

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The principal goal of the present note is to provide the proof of the main theorem of [2]. The notation of [2] will be followed faithfully with only a minimum amount of recall. Let Q be a stable picture of a polyhedron $\boldsymbol{P}$ with S denoting the set of vertices of $\boldsymbol{P}$. Our target is the asymptotic behavior of

(1) $$a_r^Q=\sum_{\vec{m}\in A(Q),|\vec{m}|=r} f(\vec{m}),$$

where

(2) $$f(\vec{x})=\prod_{\alpha\in S}\frac{2\Gamma(x^F(\alpha))}{\Gamma\left(\frac{x^F(\alpha)}{2}\right)\Gamma\left(\frac{x^F(\alpha)}{2}+2\right)}\cdot\prod_{1\le i\le k}\frac{1}{\Gamma(x_i+1)}.$$

Two functions of a positive real variable r will be said to be *asymptotically equal* if their ratio approaches 1 as $r\to\infty$. In a separate paper, we will study the generating function defined by a_r as well as the differential equations satisfied by these generating functions. The numbering of the sections present note will continue that of [2].

§5. **Gamma function estimates.** The purpose of this section is to gather some lemmas in preparation for the following section. We will use the following three results. The first two can be found in Artin [1]. The third is a calculus exercise.

(5.1) *Stirling's Formula,*

$$n!=n^n\cdot e^{-n}\cdot(2\pi n)^{1/2}\cdot(1+O(n^{-1}))\qquad \text{as } n\longrightarrow\infty.$$

$$\Gamma(x)=(2\pi)^{1/2}\cdot x^{x-1/2}\cdot e^{-x}\cdot(1+O(x^{-1}))\qquad \text{as } x\longrightarrow\infty.$$

(5.2) Let C denote Euler's constant. Then,

$$\frac{d}{dx}\log(\Gamma(x))=-C-\frac{1}{x}+\sum_{n\ge1}\left\{\frac{1}{n}-\frac{1}{n+x}\right\}.$$

(5.3) Let $g\colon[\frac{1}{2},\infty)\to(0,\infty)$ be a function with $g'(t)<0$, $g''(t)>0$ for $t\ge1/2$. Suppose further that $\int_{1/2}^{\infty}g(t)dt<\infty$. Then,

$$0<\int_{1/2}^{\infty}g(t)dt-\sum_{n\ge1}g(n)<\{g(1/2)-g(1)\}/4.$$

From (5.1) and the functional equation of Γ, we obtain

(5.4) As $x\to\infty$, we have

$$\frac{2\Gamma(x)}{\Gamma\left(\frac{x}{2}\right)\cdot\Gamma\left(\frac{x}{2}+2\right)}=\frac{4\cdot2^x}{(2\pi)^{1/2}\cdot x^{3/2}}\{1+O(x^{-1})\}.$$

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