# 54. A Generalization of the Chowla-Selberg Formula and the Zeta Functions of Quadratic Orders

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The purpose of this note is to present an identity which generalizes a formula of Chowla and Selberg on the periods of CM elliptic curves in connection with the zeta functions of imaginary quadratic orders. We first proposed the identity as the numerical evidence and then Y. Nakkajima and Y. Taguchi have proved it algebraically by using the technique from arithmetic geometry ([2]). We employ here an analytical approach using zeta functions.

**1. Review and result.** Let $K$ be an imaginary quadratic field with discriminant $D$, $O_K$ its ring of integers, $w$ the order of unit group $O_K^{\times}$, and $\chi$ the quadratic Dirichlet character modulo $|D|$ determined by the extension $K/\boldsymbol{Q}$. For a lattice $L$ in $\boldsymbol{C}$, define

$$g_2(L)=60\sum_{\lambda\in L}{}'\lambda^{-4},\ g_3(L)=140\sum_{\lambda\in L}{}'\lambda^{-6}$$

*and*

$$\Delta(L)=g_2(L)^3-27g_3(L)^2.$$

The "discriminant" $\Delta(L)$ is non-zero and has the property

$$(1)\qquad \Delta(\alpha L)=\alpha^{-12}\Delta(L)\qquad \textit{for } \alpha\in\boldsymbol{C}^{\times}.$$

Take an ideal $\mathfrak{A}$ of $K$ and consider the value

$$F(\mathfrak{A})=\Delta(\mathfrak{A})\Delta(\mathfrak{A}^{-1}).$$

By (1) it depends only on the class of $\mathfrak{A}$ in the ideal class group $Cl(O_K)$. Any period of elliptic curves with complex multiplication in $K$ differs by an algebraic constant from the 24-th root of $F(\mathfrak{A})$. The formula of Chowla-Selberg expresses the product of $F(\mathfrak{A})$ over $Cl(O_K)$ by gamma values:

$$(2)\qquad \prod_{Cl(O_K)}F(\mathfrak{A})=\left(\frac{2\pi}{|D|}\right)^{12h}\prod_{a=1}^{|D|-1}\Gamma\left(\frac{a}{|D|}\right)^{6w\chi(a)}.$$

Now let $O_f$ be an order of $K$ of conductor $f$. We denote by $Cl(O_f)$ the group of proper $O_f$-ideal classes and by $h_f$ its order (the class number of $O_f$). The function $F(\mathfrak{A})$ is defined also on the proper $O_f$-ideals $\mathfrak{A}$ and again depends only on the class in $Cl(O_f)$. Y. Nakkajima and Y. Taguchi ([2]), inspired by K. Fujiwara, gave an algebraic proof (assuming (2), though) of the following

**Theorem.** *We have*

$$(3)\qquad \prod_{Cl(O_f)}F(\mathfrak{A})=\left(\frac{2\pi}{|f^2D|}\right)^{12h_f}\left(\prod_{a=1}^{|D|-1}\Gamma\left(\frac{a}{|D|}\right)^{6w\chi(a)}\right)^{h_f/h}\times\prod_{p|f}p^{12e(p)},$$

*where*