49. Note on Isometry Invariant Geodesic on Two Dimensional Spherical Manifold*),**)

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(Communicated by Kunihiko KODAIRA, M. J. A., Sept. 12, 1990)

Let f be an isometry on a Riemannian manifold. Then a geodesic c is called "f-invariant" (or isometry invariant geodesic) if $f(c(t))=c(t+1)$ for any $t \in R$. In general such a geodesic is not necessary closed except isometry of finite order. (We studied the structure of such a geodesic for an isometry of finite order [2], [3].) It is very interesting to ask what isometry has an invariant closed geodesic. There is no general information about this problem other than the following theorem: *if there is only a finite number of geodesics, then it is closed* [1] (cf. [5]). In this note we assert that any isometry of small displacement on two dimensional spherical manifold has an invariant closed geodesic. Though our result can be proved by using Theorem 3.2 [1], we give here directly another proof because our method is very elementary and more geometrical.

An isometry f on a Riemannian manifold M is called "small displacement" if for each $x \in M$ there is a unique minimizing geodesic from x to $f(x)$. Our main result is the following

Theorem. *Let M be a Riemannian manifold homeomorphic to S^2. Let f be a small displacement. Then there is a closed f-invariant geodesic which is not a point curve.*

The real valued function δ_f on M is defined by $\delta_f(x)=d(x, f(x))$ where d is a distance function of M and f is an isometry. In [4] Ozols has studied the critical point of δ_f. Let $\mathrm{Crt}(f)$ be a set of critical point of δ_f^2, then $\mathrm{Crt}(f)=F(f) \cup$ (critical point of δ_f on $(M-F(f))$ where $F(f)$ is a set of a fixed point of f.

Fact. *Let f be a small displacement. Then $x \in Crt(f)-F(f)$ if and only if f preserves the minimizing geodesic from x to $f(x)$.*

If M is compact, then δ_f^2 has a maximum point on M. Thus we have

Lemma 1. *Let f be a small displacement on a compact manifold. Then there exists a f-invariant geodesic which is not a point curve.*

By this lemma we have only to prove the theorem when our isometry is not finite order. From now on we assume that Riemannian manifold is homeomorphic to S^2 and f is a small displacement of which order is not finite. Since f is a small displacement, f is homeomorphic to 1 and so

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*) Dedicated to Prof. Akio HATTORI on his 60th birthday.

**) This research was partially supported by Grant-in-Aid for Scientific Research (No. 58540018), Ministry of Education, Science and Culture of Japan.