

24. Notes on Certain Analytic Functions

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1. Introduction. Let $\mathcal{A}(n)$ denote the class of functions of the form

$$(1.1) \quad f(z) = z + \sum_{k=n+1}^{\infty} a_k z^k \quad (n \in \mathcal{N} = \{1, 2, 3, \dots\})$$

which are analytic in the unit disk $\mathcal{U} = \{z : |z| < 1\}$.

A function $f(z)$ belonging to the class $\mathcal{A}(1)$ is said to be starlike with respect to the origin if it satisfies

$$(1.2) \quad \operatorname{Re} \left\{ \frac{zf'(z)}{f(z)} \right\} > 0 \quad (z \in \mathcal{U}),$$

which is equivalent to

$$(1.3) \quad \left| \arg \left(\frac{zf'(z)}{f(z)} \right) \right| < \frac{\pi}{2} \quad (z \in \mathcal{U}).$$

Let $\mathcal{S}^*(\alpha)$ be the subclass of $\mathcal{A}(1)$ consisting of functions which satisfy

$$(1.4) \quad \left| \arg \left(\frac{zf'(z)}{f(z)} \right) \right| < \frac{\pi}{2} \alpha$$

for some α ($0 < \alpha \leq 1$) and for all $z \in \mathcal{U}$. Clearly, a function $f(z)$ belonging to the class $\mathcal{S}^*(\alpha)$ is starlike with respect to the origin in the unit disk \mathcal{U} .

Further, a function $f(z)$ in the class $\mathcal{A}(1)$ is said to be convex of order α if it satisfies

$$(1.5) \quad \operatorname{Re} \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} > \alpha$$

for some α ($0 \leq \alpha < 1$) and for all $z \in \mathcal{U}$. We denote by $\mathcal{K}(\alpha)$ the subclass of $\mathcal{A}(1)$ consisting of all such functions.

2. Some properties. We begin with the statement of the following lemma due to Miller and Mocanu [1].

Lemma 1. Let $f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \dots$ ($n \in \mathcal{N}$) be analytic in \mathcal{U} with $f(z) \neq a$. If $z_0 = r_0 e^{i\theta_0}$ ($0 < r_0 < 1$) and

$$|f(z_0)| = \max_{|z| \leq r_0} |f(z)|,$$

then

$$(2.1) \quad \frac{z_0 f'(z_0)}{f(z_0)} = m$$

and

$$(2.2) \quad \operatorname{Re} \left\{ 1 + \frac{z_0 f''(z_0)}{f'(z_0)} \right\} \geq m,$$

where $m \geq 1$ and

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