22. *Convex Operators and Convex Integrands*

By Naoto KOMURO
Department of Mathematics, Hokkaido Educational University Asahikawa

(Communicated by Kôsaku YOSIDA, M. J. A., March 13, 1989)

Introduction. Let Ω be an arbitrary set, let Σ be a σ-field of subsets of Ω (the measurable sets), and let μ denote a nonnegative σ-finite measure on Σ. Let $S(\Omega)$ be the space of all finite valued measurable functions on Ω. We identify f and $g \in S(\Omega)$ if they differ only on a set of μ-measure zero. With the usual ordering $S(\Omega)$ is an order complete vector lattice. A mapping $F: \boldsymbol{R}^d \supset D(F) \to S(\Omega)$ is called a convex operator if $D(F)$ the domain of F is a convex set of the d-dimensional Euclidean space $\boldsymbol{R}^d$ and

$$F(\lambda x+(1-\lambda)y) \leqq \lambda F(x)+(1-\lambda)F(y)$$

holds for every $x, y \in D(F)$ and $\lambda \in [0, 1]$. Many kinds of ordered vector spaces can be regarded as the subspaces of $S(\Omega)$, and hence this class of convex operators covers many cases. A function $f: \boldsymbol{R}^d \times \Omega \to \boldsymbol{R} \cup \{+\infty\}$ is said to be a *convex integrand* if $f(\cdot, t)$ is a convex function for each $t \in \Omega$. We say that a convex integrand f is a *representation* of a convex operator F if $f(x, t)$ is measurable in $t \in \Omega$ and $f(x, \cdot) = F(x)$ holds for every $x \in D(F)$. Our main result (Theorem 1) asserts that every convex operator $F: \boldsymbol{R}^d \supset D(F) \to S(\Omega)$ has at least a representation of F. In [3], one can see the proof of this result in one dimensional case. In general case, the proof is more complicated. In § 2, we consider the relations between convex operators and their representations. In § 3, we generalize the Fenchel-Moreau theorem by using representations, and give some conditions with which a convex operator can be represented by a normal convex integrand.

§ 1. Representation theorem.

Theorem 1. *For every convex operator $F: \boldsymbol{R}^d \supset D(F) \to S(\Omega)$, there exists at least a representation of F.*

Outline of the proof. The proop is done by constructing a representation. The difficulty is to determine the value of $f(x, t)$ when x belongs to $\partial D(F)$ the boundary of $D(F)$. For each $x \in \partial D(F)$, let L_x be the largest linear manifold such that some neighborhoods of x in L_x are contained by $\partial D(F)$. First we define the values of $f(x, t)$ on $D^\circ(F) \times \Omega$ by a countable argument which is an analogy of the proof in one dimensional case. Next, for each L_x with $\dim L_x = d-1$, we define $f(y, t)$ on $(L_x \cap \partial D(F)) \times \Omega$ satisfying the followings.

(a) $\sup_{z \in D(F)} \lim_{\lambda \to 0} f(y+\lambda(z-y), t) \leqq f(y, t)$ for every $y \in L_x$,

(b) $f(\cdot, t)$ is convex on L_x on $L_x \cap \partial D(F)$ for every $t \in \Omega$,

(c) $f(y, \cdot) = (F(y))(\cdot)$ for every $y \in L_x \cap D(F)$.

We can choose such values for $f(y, t)$ if we use the fact that, for each $y \in L_x$,