11. A Certain Functional Derivative Equation Corresponding to $\square u+cu+bu^2+au^3=g$ on R^{d+1}

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(Communicated by Kôsaku YOSIDA, M. J. A., Feb. 13, 1989)

Introduction and results. L_r^p $(1\leq p\leq\infty, r\in R)$ denotes the space of weighted p-summable functions on R^d with norm given by $|u|_{p,r}=\left(\int_{R^d}(1+|x|^2)^{rp/2}|u(x)|^p dx\right)^{1/p}$ or $|u|_{\infty,r}=\text{ess.}\sup_{x\in R^d}(1+|x|^2)^{r/2}|u(x)|$. When $r=0$, we put $L^p=L_0^p$ with $|u|_p=|u|_{p,0}$. For $s\in N$, $\|u\|_{s,r}=\left(\int_{R^d}(1+|x|^2)^r \sum_{|\alpha|\leq s}|D^\alpha u(x)|^2 dx\right)^{1/2}$ represents the norm of H_r^s, the weighted Sobolev space of order s on R^d. For general $s\in R$, H_r^s is defined by using the interpolation theory and H^s stands for H_0^s with $\|u\|_s=\|u\|_{s,0}$. The dual space of L_r^p is L_{-r}^q for $1\leq p<\infty$ with $1/p+1/q=1$. $H_{-r}^{-s}=(\mathring{H}_r^s)^*$ for $s\geq 0$ with $\mathring{H}_r^s=\mathring{H}_r^s(R^d)$ $(s\geq 0)$ being the closure of $C_0^\infty(R^d)$ in H_r^s.

Now, we put $X={}^t(V\times L^2)$ and $X^*=V^*\times L^2$ with norms $\|U\|_X=\|u\|_V+|v|_2$ and $\|\Xi\|_{X^*}=\|\xi\|_{V^*}+|\eta|_2$ for $U={}^t(u,v)$ and $\Xi=(\xi,\eta)$. Here, $V=H^1\cap L^4$ and $V^*=H^{-1}+L^{4/3}$ with norms $\|u\|_V=\|u\|_1+|u|_4$ and $\|\xi\|_{V^*}=\inf_{\xi=\xi_1+\xi_2}(\|\xi_1\|_{-1}+|\xi_2|_{4/3})$.

Our aim of this paper is to solve the following problems: Let $0<T_0\leq\infty$.

(I) Find a functional $W(t,\Xi)$ on $t\in(0,T_0)\times X^*$ satisfying

$$\text{(I.1)}\quad \frac{\partial}{\partial t}W(t,\Xi)=\int_{R^d}\left[\eta(x)\left((\Delta-c)\frac{\delta W(t,\Xi)}{\delta\xi(x)}+ib\frac{\delta^2 W(t,\Xi)}{\delta\xi(x)^2}+a\frac{\delta^3 W(t,\Xi)}{\delta\xi(x)^3}\right)+\xi(x)\frac{\delta W(t,\Xi)}{\delta\eta(x)}+i\eta(x)g(x,t)W(t,\Xi)\right]dx,$$

$$\text{(I.2)}\quad W(t,0)=1,\qquad W(0,\Xi)=W_0(\Xi).$$

Here given data are $W_0(\Xi)$ and $g(x,t)$.

(II) Find a family of Borel measures $\{\mu(t,dU)\}_{0<t<T_0}$ on X satisfying

$$\text{(II)}\quad \int_0^{T_0}\int_X\frac{\partial\Phi(t,U)}{\partial t}\mu(t,dU)dt+\int_X\Phi(0,U)\mu_0(dU)$$
$$=-\int_0^{T_0}\int_X\int_{R^d}\left[(\Delta u(x)-f(u(x))+g(x,t))\frac{\delta\Phi(t,U)}{\delta v(x)}+v(x)\frac{\delta\Phi(t,U)}{\delta u(x)}\right]\times dx\mu(t,dU)dt$$

for suitable 'test functionals' $\Phi(t,U)$ with given data $\mu_0(dU)$ and $g(x,t)$.

For the notational simplicity, we put here $f(u)=au^3+bu^2+cu$, $F(u)=au^4/4+bu^3/3+cu^2/2$ and

$$H(U)=H(u,v)=\int_{R^d}\{|v(x)|^2/2+|\nabla u(x)|^2/2+F(u(x))\}dx.$$