89. Some Remarks on Index and Entropy for von Neumann Subalgebras

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In the present note, we introduce two notions, i.e. *finite type* of inclusion relation of von Neumann algebras and *indicial derivative*. The former is a generalization of index finite type and entropy finite type. The latter is a substitute of the index initiated by V. Jones [3] and extended by H. Kosaki [6]. The aim of the present note is to report that the indicial derivative produces both of the index and Pimsner-Popa's entropy [7].

1. Let $M \supset N$ be a pair of von Neumann algebras on a Hilbert space H. The representation space H is assumed to be separable throughout the present note. For the pair $M \supset N$, let $P(M,N)$ denote the set of all faithful normal semifinite N-valued weights on M. Moreover, set $P_1(M,N)=\{E \in P(M,N):\ \sigma_t^E = id\}$ and $E_1(M,N)=\{E \in P_1(M,N):\ E(1)=1\}$. $P(M,C)$ [resp. $E_1(M,C)$] is often denoted by $P(M)$ [resp. $E_1(M)$]. For each $E \in P(M,N)$, let E^c denote the restriction of E to $N' \cap M$ and let E^{-1} denote the Haagerup correspondent of E, uniquely determined by the equation of spatial derivative $\Delta((\varphi \circ E)/\psi)=\Delta(\varphi/(\psi \circ E^{-1}))$ for $\varphi \in P(N)$ and $\psi \in P(M')$. For more details, refer to [1], [2].

Lemma 1. *Let $M \supset N$ be as above. Then, there exists $E \in E_1(M,N)$ with $(E^{-1})^c \in P_1(N' \cap M, Z(M))$ if and only if $E_1(M,N) \neq \emptyset$ and $E_1(N',M') \neq \emptyset$.*

When a pair $M \supset N$ of von Neumann algebras satisfies the conditions in Lemma 1, we say that the inclusion relation $R(M,N)$ is *of finite type*. Let $ET(M,N)$ denote the set of all pairs (E,τ) where $E \in E_1(M,N)$ and $\tau \in E_1(N' \cap M)$ such that $\tau \circ E^c = \tau$. Then, if $R(M,N)$ is of finite type, $ET(M,N) \neq \emptyset$, and for each $(E,\tau) \in ET(M,N)$, one can take $E' \in E_1(N',M')$, uniquely determined by the condition that $\tau \circ (E')^c = \tau$ and we call it *standard correspondent* of E w.r.t. τ. In this case, a generalized Pedersen-Takesaki's derivative dE^{-1}/dE' is well defined by $dE^{-1}/dE' = d(\varphi \circ E^{-1})/d(\varphi \circ E')$ for $\varphi \in P(M')$ because the derivative $d(\varphi \circ E^{-1})/d(\varphi \circ E')$ does not depend on the choice of $\varphi \in P(M')$. Since this derivative dE^{-1}/dE' is determined for $(E,\tau) \in ET(M,N)$, we denote it by $I_\tau^E(M|N)$ and we call it *indicial derivative* of E w.r.t. τ.

Lemma 2. *Let $M \supset N$ be a pair of von Neumann algebras such that $R(M,N)$ is of finite type. Then, for $(E,\tau) \in ET(M,N)$, the indicial derivative $I_\tau^E(M|N)$ is a positive selfadjoint operator affiliated with the center $Z(N' \cap M)$ of $N' \cap M$ such that $I_\tau^E(M|N) = d(\tau \circ (E^{-1})^c)/d\tau \geq 1$.*

2. For a pair $M \supset N$ of von Neumann algebras and $E \in E_1(M,N)$,