78. A Nonlinear Ergodic Theorem for Asymptotically Nonexpansive Mappings in Banach Spaces

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1. Introduction. Throughout this paper X denotes a uniformly convex real Banach space and C is a closed convex subset of X. The value of $x^* \in X^*$ at $x \in X$ will be denoted by (x, x^*). The duality mapping J (multi-valued) from X into X^* will be defined by $J(x)=\{x^* \in X^* : (x, x^*)=\|x\|^2=\|x^*\|^2\}$ for $x \in X$. We say that X is (F) if the norm of X is Fréchet differentiable, i.e., for each $x \in X$ with $x \neq 0$, $\lim_{t\to 0} t^{-1}(\|x+ty\|-\|x\|)$ exists uniformly in $y \in B_1$, where $B_r=\{z \in X : \|z\| \leqq r\}$ for $r>0$. A mapping $T: C \to C$ is said to be *asymptotically nonexpansive* if for each $n=1, 2, \cdots$

(1.1) $\|T^n x - T^n y\| \leqq (1+\alpha_n)\|x-y\|$ for any $x, y \in C$,

where $\lim_{n\to\infty} \alpha_n = 0$. In particular, if $\alpha_n=0$ for all $n \geqq 1$, T is said to be nonexpansive. The set of fixed points of T will be denoted by $F(T)$.

Throughout the rest of this paper let $T: C \to C$ be an asymptotically nonexpansive mapping satisfying (1.1).

A sequence $\{x_n\}_{n\geqq 0}$ in C is called an *almost-orbit* of T if

$$\lim_{n\to\infty} [\sup_{m\geqq 0} \|x_{n+m} - T^m x_n\|] = 0.$$

A sequence $\{z_n\}$ in X is said to be *weakly almost convergent* to $z \in X$ if

$$w-\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} z_{k+i} = z$$

uniformly in $i \geqq 0$.

The purpose of this paper is to prove the following (nonlinear) mean ergodic theorem which is an extension of [3, Theorem 1] and [1, Corollary 2.1].

Theorem. *Let $\{x_n\}_{n\geqq 0}$ be an almost-orbit of T. If X is (F) and C is bounded, then $\{x_n\}$ is weakly almost convergent to the unique point of $F(T) \cap \mathrm{clco}\, \omega_w(\{x_n\})$, where $\omega_w(\{x_n\})$ denotes the set of weak subsequential limits of $\{x_n\}$, and $\mathrm{clco}\, E$ is the closed convex hull of E.*

2. Proof of Theorem. Throughout this section, we assume C is bounded. By Bruck's inequality [2, Theorem 2.1], we get

Lemma 1. *There exists a strictly increasing, continuous, convex function $\gamma: [0, \infty) \to [0, \infty)$ with $\gamma(0)=0$ such that*

$$\left\| T^k\left(\sum_{i=1}^n \lambda_i x_i\right) - \sum_{i=1}^n \lambda_i T^k x_i \right\|$$
$$\leqq (1+\alpha_k)\gamma^{-1}\left(\max_{1\leqq i,j\leqq n}\left[\|x_i - x_j\| - \frac{1}{1+\alpha_k}\|T^k x_i - T^k x_j\|\right]\right)$$