57. On the Deift-Trubowitz Trace Formula for the 1-dimensional Schrödinger Operator

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1. Introduction. The purpose of the present work is to prove the Deift-Trubowitz trace formula

$$2i\pi^{-1}\int_{-\infty}^{\infty}\xi r_{\pm}(\xi\,;u)f_{\pm}(x,\xi\,;u)^2 d\xi=u(x) \tag{1}$$

for the 1-dimensional Schrödinger operator $H(u)=-\partial^2+u(x)$ with $u(x)\in\Pi_0$ such that $u', u''\in L^1(\boldsymbol{R})$, $H(u)$ has no bound states and satisfies the following conditions (A), (B) and (C):

(A) $r_{\pm}(\xi\,;u)=1+i\alpha_{\pm}\xi+o(\xi)$ as $\xi\to 0$ for some $\alpha_{\pm}\in\boldsymbol{R}$.

(B) $R_{\pm}(x)$, the Fourier transforms of $r_{\pm}(\xi\,;u)$, are absolutely continuous, and

$$\pm\int_{\alpha}^{\pm\infty}(1+x^2)|R'_{\pm}(x)|dx<\infty \qquad \text{for all } \alpha\in\boldsymbol{R}.$$

(C) $S_+(u)\cup S_-(u)\neq\varnothing$.

The notations used in the above are as follows:

$$\Pi_k=\{u\,|\,\text{real, continuous, }\lim_{|x|\to\infty}u(x)=0,\text{ and }|x|^k u(x)\in L^1(\boldsymbol{R})\},\qquad k\in[0,\infty),$$

$f_{\pm}(x,\xi\,;u)$ are the Jost solutions for $H(u)$, i.e., those solutions of

$$H(u)f=-f''+u(x)f=\xi^2 f,\qquad \xi\in\boldsymbol{R}\backslash\{0\} \tag{2}$$

which behave like $\exp(\pm i\xi x)$ as $x\to\pm\infty$ respectively, $r_{\pm}(\xi\,;u)$ are the reflection coefficients of $H(u)$, and $S_{\pm}(u)$ are the sets of solutions $f(x)$ of (2) for $\xi=0$ such that $\lim_{x\to\pm\infty}f(x)$ exist and belong to $(0,\infty)$, respectively. Refer [2] and [3] for detail of the scattering theory of $H(u)$ with $u\in\Pi_1$ and $u\in\Pi_0$ respectively.

The trace formula (1) was first proved by Deift and Trubowitz in [2] for the potential $u(x)$ in Π_1 with $u', u''\in L^1(\boldsymbol{R})$ such that $H(u)$ has no bound states. See also [1]. Our aim is to extend the formula (1) to the potential mentioned above.

2. Darboux transformation. Let $P(H(u))$ be the set of positive solutions of the equation (2) for $\xi=0$. Suppose $P(H(u))\neq\varnothing$. Put $A_g=g^{-1}\partial g$ for $g\in P(H(u))$. Then $H(u)=A_gA_g^*$ follows, where A_g^* is the formal adjoint of A_g. We call $H^*(u\,;g)=A_g^*A_g$ the Darboux transformation of $H(u)$ by $g(x)$. Put

$$u^*(x\,;g)=u(x)-2(\log g(x))'',$$

then $H^*(u\,;g)=-\partial^2+u^*(x\,;g)$ follows.

Let $\Lambda^{(k)}$, $k\geqq 2$, be the set of potentials $u(x)\in\Pi_k$ such that $H(u)$ has no