52. A Note on Capitulation Problem for Number Fields. II

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In the present note, we shall again consider a capitulation problem for number fields which we discussed in our earlier paper [2]. Using some properties of $\boldsymbol{Z}_p$-extensions of number fields, we shall prove the following:

Proposition. *For each prime number $p \geqslant 2$, there exist infinitely many finite algebraic number fields k such that the p-class group of k capitulates in a proper subfield of Hilbert's p-class field over k.*

We note that in the special case $p=2$, the proposition was proved in [2] by elementary argument.

1. Let M be any number field, finite or infinite over the rational field $\boldsymbol{Q}$. Throughout the following, we fix a prime number $p \geqslant 2$ and denote by $A(M)$ the p-primary component of the ideal class group of M; if M is finite over $\boldsymbol{Q}$, this is the p-class group of M, denoted by $C_{M,p}$ in [2].

Lemma 1. *Let k' be an unramified cyclic extension of degree p over a finite algebraic number field k. Then $A(k)$ capitulates in k' if and only if the following* a), b) *hold*:

a) *there exists a prime ideal of k which is undecomposed and principal in k',*

b) *if the class of an ideal $\mathfrak{a}'$ of k' belongs to $A(k')$, the norm $N_{k'/k}(\mathfrak{a}')$ is a principal ideal in k'.*

Proof. Let K and K' denote Hilbert's p-class fields over k and k' respectively: $k \subseteq k' \subseteq K \subseteq K'$. Let t: $\mathrm{Gal}(K/k) \to \mathrm{Gal}(K'/k')$ be the transfer map. Fix an element σ of $\mathrm{Gal}(K'/k)$ such that the restriction $\sigma | k'$ is a generator of $\mathrm{Gal}(k'/k)$. Then, for any τ in $\mathrm{Gal}(K'/k')$, we have

$$t(\tau | K) = \prod_{i=0}^{p-1} \sigma^i \tau \sigma^{-i}.$$

By Artin [1], $A(k)$ capitulates in k' if and only if $\mathrm{Im}(t)=1$. Hence the lemma follows from the fact that a) is equivalent with $t(\sigma | K)=1$ and b) with $t(\tau | K)=1$ for all τ in $\mathrm{Gal}(K'/k')$.

2. Let $\boldsymbol{Q}_\infty$ denote the unique $\boldsymbol{Z}_p$-extension over $\boldsymbol{Q}$: $\mathrm{Gal}(\boldsymbol{Q}_\infty/\boldsymbol{Q}) \simeq \boldsymbol{Z}_p$, and let

$$\boldsymbol{Q} = \boldsymbol{Q}_0 \subset \boldsymbol{Q}_1 \subset \cdots \subset \boldsymbol{Q}_n \subset \cdots \subset \boldsymbol{Q}_\infty$$

be the sequence of intermediate fields for $\boldsymbol{Q}_\infty/\boldsymbol{Q}$. For each $n \geqslant 0$, let $\mathfrak{p}_n$ be the unique prime ideal of $\boldsymbol{Q}_n$, dividing the rational prime p; $\mathfrak{p}_n$ is a principal ideal in $\boldsymbol{Q}_n$ and $\mathfrak{p}_{n+1}^p = \mathfrak{p}_n$ for $n \geqslant 0$.

Let F be a real cyclic extension of degree p over $\boldsymbol{Q}$ such that

i) (p) is a prime ideal in F,