51. A Note on a Recent Paper on Iwasawa on the Capitulation Problem

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Introduction. Let $n\geq 1$ and let $p_1, \cdots, p_n$ be distinct primes in $N=\{z\in Z; z>0\}$, each congruent to $1 \pmod 4$. Let K_n be the quadratic field $Q(\sqrt{p_1\cdots p_n})$, and let $\mathcal{O}_n$ be the ring of algebraic integers in K_n. It is a famous unsolved problem to give simple conditions on $p_1, \cdots, p_n$ which are necessary and sufficient to ensure that $N_n(\varepsilon)=+1$ for every unit ε of $\mathcal{O}_n$. (Here N_n is the K_n/Q-norm.) Legendre in 1785 showed [3] that if $n=1$ there is always an ε in $\mathcal{O}_1$ with $N_1(\varepsilon)=-1$. However, for $n>1$, the present state of knowledge is still unsatisfactory. The aim of this note is to give a simple proof of

Theorem 1. *Let $n\geq 2$ be fixed, and let $p_1, \cdots, p_{n-1}$ be such that the Legendre symbol (p_j/p_k) equals $+1$ whenever $j\neq k$ and $j, k\leq n-1$. Then there are infinitely many choices of p_n such that $N_n(\varepsilon)=+1$ for every unit ε of $\mathcal{O}_n$.*

Theorem 1 answers a generalisation of a question raised by K. Iwasawa in a recent paper [2] on the capitulation problem. Theorem 1 is not a new result; the case $n=2$ occurs in work of A. Scholz [6], while the general case is implicit in work of L. Rédei [5], although his proof is very complicated. We should perhaps remark that the long series of papers Rédei over the years 1932–53 still contains almost all the significant known results on the signs of the $N_n(\varepsilon)$ (see [5] and the bibliography (and Chapter III) of [4]). The reader is warned that there is a serious error in the "analytical" part of [5], which the author hopes to correct in a forthcoming paper. Our proof of Theorem 1 is quite simple, relying only on standard properties of biquadratic residues in $Z[i]$ $(i=\sqrt{-1})$. For these we refer the reader to the excellent book of K. Ireland and M. Rosen [1]; all results which we state without proof are contained in the text and exercises of Chapter 9 of their book.

1. *A necessary condition for $N_n(\varepsilon)=-1$.* We retain the notation of the introduction. A number λ in $R=Z[i]$ is called *primary* if $\lambda\equiv 1 \pmod{(1+i)^3}$. If $p\in N$ is prime and $p\equiv 1 \pmod 4$ we have $p=\pi\bar{\pi}$, where π is primary and irreducible, while $\bar{\pi}$ is the complex conjugate of π. If also σ is primary irreducible and $p=\sigma\bar{\sigma}$, then $\sigma=\pi$ or $\bar{\pi}$.

If π is primary irreducible and $\alpha\in R$, $\pi\nmid\alpha$, the biquadratic residue symbol $(\alpha/\pi)_4$ is defined to be the unique power of $i=\sqrt{-1}$ such that $(\alpha/\pi)_4\equiv\alpha^{(p-1)/4} \pmod{\pi}$, where $p=\pi\bar{\pi}$ is prime in N, $p\equiv 1 \pmod 4$.