49. Compactness Criteria for an Operator Constraint in the Arkin-Levin Variational Problem

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1. Introduction. Let $(S, \mathcal{E}_S, \mu)$ and $(T, \mathcal{E}_T, \nu)$ be measure spaces and assume that a trio of functions $u: S\times T\times \boldsymbol{R}^l \to \boldsymbol{R}$, $g: S\times T\times \boldsymbol{R}^l \to \bar{\boldsymbol{R}}^k$, and $\omega: T\to \boldsymbol{R}^k$ is given. Consider the well-known Arkin-Levin variational problem formulated as follows:

$$(\boldsymbol{P})\qquad \begin{aligned} &\underset{x}{\mathit{Maximize}} \int_{S\times T} u(s,t,x(s,t))\,d(\mu\otimes\nu) \\ &\mathit{subject\ to} \\ &\int_S g(s,t,x(s,t))\,d\mu \leqq \omega(t) \quad a.e. \end{aligned}$$

The existence of optimal solutions for $(\boldsymbol{P})$ has been investigated by Arkin-Levin [1] and Maruyama [5], [6], where a special kind of infinite dimensional Ljapunov measure played a crucial role. In this paper, we shall present a more classical alternative approach to the existence problem, based upon the Continuity Theorem for nonlinear integral functionals due to Ioffe [3] and the Compactness Theorem stated and proved in the next section.

2. Compactness Theorem.

Theorem 1 (Compactness Theorem). *Let $(S, \mathcal{E}_S, \mu)$ and $(T, \mathcal{E}_T, \nu)$ be finite measure spaces and $f: S\times T\times \boldsymbol{R}^l \to \bar{\boldsymbol{R}}$ be $(\mathcal{E}_S\otimes\mathcal{E}_T\otimes\mathcal{B}(\boldsymbol{R}^l), \mathcal{B}(\bar{\boldsymbol{R}}))$-measurable, where $\mathcal{B}(\cdot)$ stands for the Borel σ-field on $(\cdot)$. We denote by $f^*(s,t,\cdot)$ the Young-Fenchel transform of $x\mapsto f(s,t,x)$ for any fixed $(s,t)\in S\times T$; i.e. $f^*(s,t,y)=\sup_x(\langle y,x\rangle - f(s,t,x))$, $y\in \boldsymbol{R}^l$. If f satisfies the growth condition:*

$$\mathrm{Dom}\int_{S\times T} |f^*(s,t,y)|\,d(\mu\otimes\nu) = \boldsymbol{R}^l;$$

$$i.e.\ \int_{S\times T} |f^*(s,t,y)|\,d(\mu\otimes\nu) < \infty \quad \text{for all } y\in \boldsymbol{R}^l,$$

then the set

$$F_c = \left\{ x\in L^1(S\times T, \boldsymbol{R}^l) \,\middle|\, \int_S f(s,t,x(s,t))\,d\mu \leqq c(t)\ \ a.e. \right\}$$

is weakly relatively compact in $L^1(S\times T, \boldsymbol{R}^l)$ for any $c\in L^1(T, \boldsymbol{R})$.

We need a lemma due to Ioffe-Tihomirov [4] (p. 358–359).

Lemma. *Let $(T, \mathcal{E}, \eta)$ be a measure space and $f: T\times \boldsymbol{R}^l \to \bar{\boldsymbol{R}}$ be a measurable function which satisfies the growth condition:*

$$\mathrm{Dom}\int_T |f^*(t,y)|\,d\eta = \boldsymbol{R}^l; \quad i.e.\ \int_T |f^*(t,y)|\,d\eta < \infty \quad \text{for all } y\in \boldsymbol{R}^l.$$