41. On the Structure and the Homology of the Torelli Group

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1. Introduction. In a series of papers [2], [3], [4], [5], Johnson developed a detailed study of the structure of the Torelli group and obtained several fundamental results. In our papers [12], [13] we have combined his method with our own [9], [10], [11] and clarified the relationship between the Casson invariant of oriented homology 3-spheres and the structure of certain subgroups of the mapping class groups of orientable surfaces. The purpose of the present note is to announce our recent results along the lines of the above mentioned works. More precisely, on the one hand we show a procedure which enhances Johnson's method of investigating the structure of the mapping class groups and on the other hand we obtain a certain information about the second homology of the Torelli group.

Details will appear elsewhere.

2. A refined from of Johnson's homomorphisms. Here we first recall the definition of Johnson's homomorphisms (see [3], [12] for details) and then we present a refined form of them. Let Σ_g be a closed oriented surface of genus $g\geq 2$ and let $\Gamma_1=\pi_1(\Sigma_g\backslash \mathrm{Int}\, D^2)$ which is a free group of rank $2g$. Define inductively $\Gamma_{k+1}=[\Gamma_k, \Gamma_1]$ $(k=1, 2, \cdots)$ and put $N_k=\Gamma_1/\Gamma_k$ which we call the k-th nilpotent quotient of Γ_1. We write H for $N_2=H_1(\Sigma_g; \boldsymbol{Z})$. It is well known that the graded Lie algebra $\oplus\Gamma_k/\Gamma_{k+1}$ is naturally isomorphic to the graded Lie algebra $\mathcal{L}=\oplus\mathcal{L}_k$ freely generated by the elements of H over $\boldsymbol{Z}$ (see [6]). Now let $\mathcal{M}_{g,1}$ be the mapping class group of Σ_g relative to an embedded disc $D^2\subset\Sigma_g$. $\mathcal{M}_{g,1}$ acts on N_k and let $\mathcal{M}(k)$ be the subgroup of $\mathcal{M}_{g,1}$ consisting of all the elements which act on N_k trivially. $\mathcal{M}(2)$ is nothing but the Torelli group $\mathcal{I}_{g,1}$. Now Johnson's homomorphism

$$\tau_k : \mathcal{M}(k) \longrightarrow \mathrm{Hom}\,(H, \mathcal{L}_k) \simeq H\otimes\mathcal{L}_k$$

is defined as $\tau_k(\varphi)(u)=[\varphi(\gamma)\gamma^{-1}]$ $(\varphi\in\mathcal{M}_{g,1}, u\in H)$, where $\gamma\in\Gamma_1$ is any element such that $[\gamma]=u$ and $[\varphi(\gamma)\gamma^{-1}]$ denotes the image in $\mathcal{L}_k$ of the element $\varphi(\gamma)\gamma^{-1}\in\Gamma_k$.

Now we describe our refined form of the homomorphism τ_k. Choose free generators $\alpha_1, \beta_1, \cdots, \alpha_g, \beta_g$ of Γ_1 such that $\pi_1(\Sigma_g)$ is obtained from Γ_1 by adding one relation $\zeta=[\alpha_1, \beta_1]\cdots[\alpha_g, \beta_g]=id$. Also choose a 2-chain $\sigma_0\in C_2(\Gamma_1)$ of the group Γ_1 such that $\partial\sigma_0=-(\zeta)$ and that its image in $C_2(\pi_1(\Sigma_g))$ is a fundamental cycle. Now let $\varphi\in\mathcal{M}(k)$. Then $\sigma_0-\varphi_*(\sigma_0)$ is a 2-cycle of the free group Γ_1 so that there exists a 3-chain $c_\varphi\in C_3(\Gamma_1)$ such that $\partial c_\varphi=\sigma_0-\varphi_*(\sigma_0)$ (we can construct such a 3-chain explicitly by making use of the