40. A Spectral Decomposition of the Product of Four Zeta-values

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The aim of this note is to inject Kuznecov's trace formula [2] into the argument of our former work [4], and to make a preparation which will be needed in our plan of a finer study of the fourth power moment of the Riemann zeta-function over the critical line.

We consider the product $\zeta(u)\zeta(v)\zeta(w)\zeta(z)$. In the region of absolute convergence this is decomposed into three parts:

$$\{\sum_{km=ln}+\sum_{km<ln}+\sum_{km>ln}\}k^{-u}l^{-v}m^{-w}n^{-z}.$$

The first sum can be computed by means of Ramanujan's identity. Let us denote the second sum by $g(u,v,w,z)$; then the third is $g(v,u,z,w)$. We put

$$\begin{aligned} g^*(u,v,w,z)=&g(u,v,w,z)-\Gamma(z)^{-1}\Gamma(1-w)\Gamma(w+z-1)\zeta(u+v)\\ &\times\zeta(w+z-1)\zeta(u-w+1)\zeta(v-z+1)\{\zeta(u+v-w-z+2)\}^{-1}\\ &-2(2\pi)^{w-u}\cos\left(\frac{\pi}{2}(u-w)\right)\Gamma(z)^{-1}\Gamma(u-w)\Gamma(1-u)\Gamma(u+z-1)\\ &\times\zeta(u+z-1)\zeta(v+w)\zeta(u-w)\zeta(v-z+1)\{\zeta(v+w-u-z+2)\}^{-1}. \end{aligned}$$

Then we are going to show that an analytic continuation of g^* can be described in terms of sums of products of Hecke L-series.

To state our result we have to introduce some terminologies from the theory of automorphic functions: Let $\{\chi_j^2+(1/4);\chi_j>0\}\cup\{0\}$ be the discrete spectrum of the non-Euclidean Laplacian acting on the usual Hilbert space of L^2 automorphic functions with respect to the full modular group. Let φ_j be the Maass wave form attached to χ_j. With the first Fourier coefficient $\rho_j(1)$ of φ_j we put $\alpha_j=|\rho_j(1)|^2(\cos(i\pi\chi_j))^{-1}$. Also, H_j is the Hecke L-series corresponding to φ_j, and ε_j is the parity sign of φ_j. Next, let $\{\varphi_{j,2k};1\leqq j\leqq d_{2k}\}$ be the orthonormal base, consisting of eigen functions of Hecke operators $T_{2k}(n)$, of the usual unitary space of regular cusp forms of weight $2k$ with respect to the full modular group. With the first Fourier coefficient $\rho_{j,2k}(1)$ of $\varphi_{j,2k}$ we put $\alpha_{j,2k}=(4\pi)^{1-2k}(2k-1)!\,|\rho_{j,2k}(1)|^2$. Finally, $H_{j,2k}$ is the Hecke L-series corresponding to $\varphi_{j,2k}$.

Further, let $\theta>1$ be a parameter, and let A_θ be the domain

$$\left\{(u,v,w,z);\,2\mathrm{Re}(z)>\mathrm{Re}(u+v+w+z)>\frac{3}{2}+2\theta,\,\mathrm{Re}(u+z)<\theta,\,\mathrm{Re}(w+z)<\theta\right\}.$$

In A_θ we define two functions Ψ_θ and Φ_θ by