38. A Discrepancy Problem with Applications to Linear Recurrences. I

By Péter KISS*),†) and Robert F. TICHY**)

(Communicated by Shokichi IYANAGA, M. J. A., May 12, 1989)

1. **Introduction.** Let $R=\{R_n\}_{n=0}^{\infty}$ be a second order linear recursive sequence of rational integers defined by

$$R_n=A\cdot R_{n-1}+B\cdot R_{n-2}\ (n>1),$$

where the initial values R_0, R_1 and A, B are fixed integers. We suppose that $AB\neq 0$, $R_0^2+R_1^2\neq 0$ and $D=A^2+4B\neq 0$. It is well-known that the terms of R can be expressed as

$$R_n=a\cdot\alpha^n-b\cdot\beta^n \tag{1}$$

for any $n\geqq 0$, where α and β are the roots of the polynomial x^2-Ax-B and

$$a=\frac{R_1-R_0\beta}{\alpha-\beta},\qquad b=\frac{R_1-R_0\alpha}{\alpha-\beta}.$$

Throughout this paper, we assume $|\alpha|\geqq|\beta|$ and that the sequence is non-degenerate, i.e. α/β is not a root of unity. We may also suppose that $R_n\neq 0$ for $n>0$ since in [2] it was proved that a non-degenerate sequence R has at most one zero term and after a movement of indices this condition will be fulfilled.

If $D=A^2+4B>0$, i.e. if α and β are real numbers, then $|\alpha|>|\beta|$ and $(\beta/\alpha)^n\to 0$ as $n\to\infty$; hence we obtain by (1)

$$\lim_{n\to\infty}(R_{n+1}/R_n)=\alpha. \tag{2}$$

The following interesting problem arises: what is the quality of approximation of α by rationals of the form R_{n+1}/R_n? In the case $D>0$ we know that there are constants $q>0$ and k_0 ($0<k_0\leqq 2$), depending on the parameters of the sequence R, such that

$$\left|\alpha-\frac{R_{n+1}}{R_n}\right|<q\cdot R_n^{-k} \tag{3}$$

for infinitely many n and for any $k\leqq k_0$, but (3) holds only for finitely many n if $k>k_0$ (see [5]). For the sequence R with initial values $R_0=0$, $R_1=1$ it was proved in [3] that $k_0=2$ if and only if $|B|=1$; furthermore

$$\left|\alpha-\frac{R_{n+1}}{R_n}\right|<\frac{1}{\sqrt{D}\cdot R_n^2}$$

for infinitely many n, and these rational numbers R_{n+1}/R_n give the best

†) Research partially supported by Hungarian National Foundation for Scientific Research Grant no. 273.

*) Teachers' Training College, Department of Mathematics, Eger, Hungary.

**) Department of Technical Mathematics, Technical University of Vienna, Vienna, Austria.