1. *Nonlinear Eigenvalue Problem $\Delta u+\lambda e^{u}=0$ on Simply Connected Domains in R^2*

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§1. **Introduction and results.** In the previous work [3, 4], we studied the connectivity of the branch of minimal solutions $\underline{C}$ starting from $(\lambda, u)=(0,0)$ and that of Weston-Moseley's large solutions C^* as $\lambda\downarrow 0$ ([6, 2]) for the nonlinear eigenvalue problem

(1.1) $$-\Delta u=\lambda e^{u} \text{ (in } \Omega) \quad\text{and}\quad u=0 \text{ (on } \partial\Omega),$$

where λ is a positive constant, $\Omega\subset R^2$ is a simply-connected bounded domain with smooth boundary $\partial\Omega$, and $u\in C^2(\Omega)\cap C^0(\bar{\Omega})$ is a classical solution. We have established the connectivity of $\underline{C}$ and C^* when Ω is close to a disc. In this note, we shall refine the result and give an explicit criterion for Ω to have such a property for (1.1).

Our basic idea was to parametrize the solutions $h={}^{T}(u,\lambda)$ of (1.1) through $s=\lambda\int_{\Omega}e^{u}dx$. Thus we introduce the nonlinear mapping $\Phi=\Phi(h,s)\colon \hat{X}\times R\to\hat{Y}$ by $\Phi(h,s)={}^{T}(\Delta u+\lambda e^{u}, \int_{\Omega}e^{u}dx-(s/\lambda))$ for $h={}^{T}(u,\lambda)$ and $s\in R_+$, where $\hat{X}={}^{T}(X\times R_+)$ and $\hat{Y}={}^{T}(Y\times R)$ with $X=C_0^{2+\alpha}(\bar{\Omega})\equiv\{v\in C^{2+\alpha}(\bar{\Omega})\mid v=0 \text{ on } \partial\Omega\}$ and $Y=C^{\alpha}(\bar{\Omega})$ for $0<\alpha<1$. For this mapping we claim that

Theorem 1. *For each zero-point (h,s) of Φ, the linearized operator $d_h\Phi(h,s)\colon\hat{X}\to\hat{Y}$ is invertible provided that $0<s<8\pi$.*

Since the a priori estimates $\|u\|_{C^0(\bar{\Omega})}\leqq -2\log(1-(s/8\pi))$ and $s|\Omega|^{-1}\exp(\|u\|_{C^0(\bar{\Omega})})\leqq\lambda\leqq\bar{\lambda}$ hold if $0<s<8\pi$ for some $\bar{\lambda}=\bar{\lambda}(\Omega)$, the first part of the following theorem follows immediately from the above one. On the other hand the latter part holds by the fact that $s<4\pi$ and $s<8\pi$ imply $\mu_1(p)>0$ and $\mu_2(p)>0$, respectively, where $\{\mu_j(p)\}_{j=1}^{\infty}(-\infty<\mu_1(p)<\mu_2(p)\leqq\cdots\to\infty)$ are the eigenvalues of $A_p\equiv -\Delta-p$ under Dirichlet condition for $p=\lambda e^{u}$:

Theorem 2. *In $s-h$ plane, there exists a branch $\mathcal{S}$ of zero-points of Φ starting from $(s,h)=(0,0)$ and continuing up to $s=8\pi$ without bending, and furthermore, there is no other zero-point of Φ other than $\mathcal{S}$ in the area $0<s<8\pi$. The corresponding branch C in $\lambda-u$ plane to $\mathcal{S}$ starts from $(\lambda,u)=(0,0)$ and bends at most once.*

On the other hand, along the Weston-Moselely's branch C^* of large solutions, we have from [4] that $S\equiv\lambda\int_{\Omega}e^{u}dx=8\pi+c\pi\lambda+0(\lambda)$ as $\lambda\downarrow 0$, where $C=C(\Omega)=-|a_1|^2+\sum_{n=3}^{\infty}(n^2/(n-2))|a_n|^2$ for the normalized Riemann mapping

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