3. On the First Eigenvalue of Some Quasilinear Elliptic Equations

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1. Introduction. Let Ω be a bounded domain in $\boldsymbol{R}^N$ with smooth boundary $\partial\Omega$. For given $p \in (1, +\infty)$, $a \in L^\infty_+(\Omega)=\{f \in L^\infty(\Omega)\,;\ f(x)\geq 0$ a.e. $x \in \Omega\}$ and $b \in L^\infty_0(\Omega)=\{f \in L^\infty(\Omega)\,;\ f^+(\cdot)=\max(f(\cdot), 0)\not\equiv 0\}$, we consider the following eigenvalue problem:

$$(E)_\lambda \quad \begin{cases} (1) & -\Delta_p u(x)+a(x)|u|^{p-2}u(x)=\lambda b(x)|u|^{p-2}u(x), \quad x \in \Omega, \quad \lambda>0, \\ (2) & u(x)=0, \quad x \in \partial\Omega, \end{cases}$$

where $\Delta_p u(x)=\operatorname{div}(|\nabla u|^{p-2}\nabla u(x))$.

The main purpose of this paper is to show that there exists a positive number λ_1, *the first eigenvalue*, such that $(E)_\lambda$ admits a positive solution if and only if $\lambda=\lambda_1$ and that λ_1 is simple, i.e., solutions of $(E)_{\lambda_1}$ forms a one dimensional subspace of $W^{1,p}_0(\Omega)$. Here u is said to be a solution of $(E)_\lambda$ if u belongs to $W^{1,p}_0(\Omega)$ and satisfies (1) in the sense of distribution. For the case where $a\equiv 0$ and $b\equiv 1$, the simplicity of λ_1 has been shown under some additional assumptions. When $N=1$, it is shown in [2] that all eigenvalues λ_k ($k \in \boldsymbol{N}$) are simple and that all eigenfunctions u_k associated with λ_k have $(k-1)$ isolated zeros in Ω. If Ω is a ball, DeThélin [5] showed the simplicity of λ_1 in the class of radially symmetric solutions by using the theory of rearrangement. Recently, Sakaguchi [4] made an argument based on a strong maximum principle to prove that λ_1 is simple provided that $\partial\Omega$ is connected. Our method of proof is quite different from those in [2], [4], [5], and requires neither the connectedness of $\partial\Omega$ nor the positivity of $b(\cdot)$.

We define $\lambda_1=\lambda_1(a, b)$ by

$$(3) \qquad 1/\lambda_1=\sup\{R(v):=B(v)/A(v)\,;\ v \in W:=W^{1,p}_0(\Omega)\setminus\{0\}\},$$

where $A(v)=\int_\Omega \{|\nabla u(x)|^p+a(x)|u(x)|^p\}dx$ and $B(v)=\int_\Omega b(x)|u(x)|^p dx$. Then our main result is stated as follows:

Theorem 1. *Eigenvalue problem $(E)_\lambda$ has a nontrivial nonnegative solution u if and only if $\lambda=\lambda_1$ and $J_{\lambda_1}(u):=A(u)-\lambda_1 B(u)=0$. Furthermore, the eigenvalue λ_1 is simple, more precisely, the set of all solutions of $(E)_{\lambda_1}$ consists of $\{tu_1\,;\ t \in \boldsymbol{R}^1\}$, where u_1 is a solution of $(E)_{\lambda_1}$ such that $u_1 \in C^{1,\theta}(\bar{\Omega})$ for some $\theta \in (0, 1)$ and $u_1(x)>0$ for all $x \in \Omega$.*

2. Some lemmas. To prove Theorem 1, we here prepare some lemmas.

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