3. On the First Eigenvalue of Some Quasilinear Elliptic Equations

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1. Introduction. Let \( \Omega \) be a bounded domain in \( \mathbb{R}^N \) with smooth boundary \( \partial \Omega \). For given \( p \in (1, +\infty) \), \( a \in L^p(\Omega) = \{ f \in L^\infty(\Omega); \ f(x) \geq 0 \ a.e. \ x \in \Omega \} \) and \( b \in L^\infty(\Omega) = \{ f \in L^\infty(\Omega); \ f^+(\cdot) = \max(f(\cdot), 0) \neq 0 \} \), we consider the following eigenvalue problem:

\[
\begin{align*}
(1) \quad & -A_\lambda u(x) + a(x)u_p = \lambda b(x)|u|^{p-2}u(x), \ x \in \Omega, \ \lambda > 0, \\
(2) \quad & u(x) = 0, \ x \in \partial \Omega,
\end{align*}
\]

where \( A_\lambda u(x) = \text{div}(\nabla |u|^{p-2} \nabla u(x)) \).

The main purpose of this paper is to show that there exists a positive number \( \lambda_1 \), the first eigenvalue, such that (1) admits a positive solution if and only if \( \lambda = \lambda_1 \) and that \( \lambda_1 \) is simple, i.e., solutions of (1) form a one dimensional subspace of \( W_0^{1,p}(\Omega) \). Here \( u \) is said to be a solution of (1) if \( u \) belongs to \( W_0^{1,p}(\Omega) \) and satisfies (1) in the sense of distribution. For the case where \( a = 0 \) and \( b = 1 \), the simplicity of \( \lambda_1 \) has been shown under some additional assumptions. When \( N = 1 \), it is shown in [2] that all eigenvalues \( \lambda_k \ (k \in \mathbb{N}) \) are simple and that all eigenfunctions \( u_k \) associated with \( \lambda_k \) have \( (k-1) \) isolated zeros in \( \Omega \). If \( \Omega \) is a ball, DeThelin [5] showed the simplicity of \( \lambda_1 \) in the class of radially symmetric solutions by using the theory of rearrangement. Recently, Sakaguchi [4] made an argument based on a strong maximum principle to prove that \( \lambda_1 \) is simple provided that \( \partial \Omega \) is connected. Our method of proof is quite different from those in [2], [4], [5], and requires neither the connectedness of \( \partial \Omega \) nor the positivity of \( b(\cdot) \).

We define \( \lambda_1 = \lambda_1(a, b) \) by

\[
\lambda_1 = \sup \{ R(v) = B(v)/A(v); \ v \in W := W_0^{1,p}(\Omega) \setminus \{0\} \},
\]

where

\[
A(v) = \int_\Omega |\nabla u(x)|^p + a(x)|u(x)|^p \, dx \quad \text{and} \quad B(v) = \int_\Omega b(x)|u(x)|^p \, dx.
\]

Then our main result is stated as follows:

Theorem 1. Eigenvalue problem (1) has a nontrivial nonnegative solution \( u \) if and only if \( \lambda = \lambda_1 \) and \( J_\lambda(u) := A(u) - \lambda B(u) = 0 \). Furthermore, the eigenvalue \( \lambda_1 \) is simple, more precisely, the set of all solutions of (1) consists of \( \{ tu_1; \ t \in \mathbb{R} \} \), where \( u_1 \) is a solution of (1) such that \( u_1 \in C^{1,\theta}(\overline{\Omega}) \) for some \( \theta \in (0, 1) \) and \( u_1(x) > 0 \) for all \( x \in \Omega \).

2. Some lemmas. To prove Theorem 1, we here prepare some lemmas.

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