106. Spectral Resolution of a Certain Summation of Series

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1. Introduction. This paper deals with the spectral resolution of a certain summation of series, the final aim being to give a method of solving recurrences involving the summation by means of its spectral decomposition. Let L denote a real linear space composed of all sequences of real numbers, and a small letter, for example, a is used to mean its element $\{a_1, a_2, \cdots\}$ $(a_i \in R)$. Our summation T_d is a linear transformation on L defined by

$$(1) \qquad T_d : a \longmapsto b, \qquad b_i = \frac{1}{d^i}\sum_{j=1}^{i}\binom{i}{j}(d-1)^{i-j}a_j \quad (i=1, 2, \cdots),$$

where d is a positive number. This summation of series is closely related to the Euler summation [1].

2. Spectral resolution of T_d. In this section, we prove that $\{T_d\}_{d>0}$ is a representation of a multiplicative group, and derive the spectral resolution with the use of its group property. Let us start by showing a lemma.

Lemma 1. *Let d_1, d_2 and d be positive numbers, and we have*

$$T_{d_1} \circ T_{d_2} = T_{d_1 d_2}, \quad T_1 = I, \quad (T_d)^{-1} = T_{1/d}.$$

Proof. Suppose that

$$b_i = \frac{1}{d_2^i}\sum_{j=1}^{i}\binom{i}{j}(d_2-1)^{i-j}a_j \quad \text{and} \quad c_k = \frac{1}{d_1^k}\sum_{i=1}^{k}\binom{k}{i}(d_1-1)^{k-i}b_i.$$

Then, a slight calculation leads to

$$c_k = \frac{1}{(d_1 d_2)^k}\sum_{j=1}^{k}\binom{k}{j}(d_1 d_2 - 1)^{k-j}a_j.$$

which proves $T_{d_1} \circ T_{d_2} = T_{d_1 d_2}$. The remaining two are obvious.

This lemma shows that each T_d is a non-singular transformation and further the family $\{T_d\}_{d>0}$ is a representation on L of a Lie group (R^+, x). Exchange the parameter d for t subject to $d = e^t$ and calculate $d/dt(T_d[a])|_{t=0}$ formally. Then, we have the formal generating operator of T_d as follows;

$$(2) \qquad -a_1\frac{\partial}{\partial a_1} + (2a_1 - 2a_2)\frac{\partial}{\partial a_2} + \cdots + (na_{n-1} - na_n)\frac{\partial}{\partial a_n} + \cdots.$$

For the time being, discussion is made on an m-dimensional linear space $\bar{L}$ which is of the first m terms $\bar{a} = \{a_1, \cdots, a_m\}$ of every element of L. It is easy to see from the definition (1) that the action of T_d can be restricted on $\bar{L}$, whose restriction we denote by $\bar{T}_d$. Then, $\bar{T}_d$ gives an R^+-action on $\bar{L}$ and its generator is expressed as a sum of first m components of (2). Since $\bar{T}_d$ is a linear transformation, it is expressed as an m-th order matrix, which is obtained by means of the generator as follows: