101. A Construction of Negatively Curved Manifolds

By Koji FUJIWARA
Department of Mathematics, University of Tokyo

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§1. Introduction. Let V be a complete Riemannian manifold with $-b<K<-a<0$ and $\mathrm{vol}(V)<\infty$. Then it is known that each end of V is an infranilmanifold ([1], [2]).

But if we change the condition $-b<K<-a<0$ to $-b<K<0$, then the conclusion does not hold in general. In this paper we will give a counter-example; if the dimension is bigger than three, there is a complete manifold V with $-b<K<0$ and $\mathrm{vol}(V)<\infty$ such that the end is not an infranilmanifold, and in the case that the dimension is three, the end is a torus.

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§2. Theorem and its proof. Theorem. *Let V be a closed manifold with $K\equiv -1$ and W a closed totally geodesic submanifold of codimension 2 in V.*

Then $V\setminus W$ admits a complete metric with $-a<K<0$ and $\mathrm{vol}(V\setminus W)<\infty$, where $a>0$.

Remark 1. A pair (V, W) with the above property exists.

Remark 2. In this theorem, the end of $V\setminus W$ is a S^1-bundle over a hyperbolic manifold W, which is not an infranilmanifold.

Proof. Let $\sigma=\mathrm{inj}(W;V)$, and take a σ-neighborhood U of W in V. We introduce a polar coordinate (w, θ, r) on U. Then $U=W\times S^1\times(0,\sigma)$ and we can write the hyperbolic metric g_V of V as follows on U ([4], [3]),

$$g_V=\cosh^2(r)g_W+\sinh^2(r)d\theta^2+dr^2 \qquad (0\le\theta\le 2\pi,\ 0\le r\le\sigma) \tag{1}$$

where g_W denotes the induced metric on W.

We are going to change the metric g_V to a new metric $h_{V'}$ on $V'=V\setminus W$ as follows. Using a positive function $f(r)$, we set

$$h_{V'}=\cosh^2(r)g_W+\sinh^2(r)d\theta^2+f^2(r)dr^2 \qquad (0\le\theta\le 2\pi,\ 0\le r\le\sigma). \tag{2}$$

To choose a suitable function $f(r)$, we compute the sectional curvature K_h of the metric $h_{V'}$. First, note that a vector field ξ on W naturally extends to a vector field on U, and we also denote it by ξ. The Riemannian connection ∇ of $h_{V'}=\langle\ ,\ \rangle$ is given as follows, where D denotes the Riemannian connection on W, and $\xi, \zeta, \cdots$ denote vector fields on W or their extentions to U.

$$\begin{cases} \nabla_\xi\zeta=D_\xi\zeta-\tanh(r)\langle\xi,\zeta\rangle\dfrac{\partial}{\partial r} \\ \nabla_\xi\dfrac{\partial}{\partial\theta}=\nabla_{\partial/\partial\theta}\xi=0 \end{cases}$$