96. On Quasi-reflexive Rings (Semigroups)

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A right ideal I of a ring (semigroup) R is called *right quasi-reflexive* [4] if whenever A and B are right ideals of R with $AB\subseteq I$ then $BA\subseteq I$. A ring (semigroup with 0) R is said to be *right quasi-reflexive* if (0) is a right quasi-reflexive ideal of R. The concept of a left *quasi-reflexive ring* (*semigroup with* 0) is defined analogously. Evidently, semiprime rings (semiprime semigroups with 0) are left and right quasi-reflexive.

In [4] we call a ring *strongly subcommutative* if every right ideal of it is right quasi-reflexive; any ring of this class of rings is therefore right quasi-reflexive.

It is the purpose of this note to extend two results of [4], Propositions 3 and 4, and Theorem 7.4 of [2] to a wider class of rings (semigroups with 0), i.e. to the class of right quasi-reflexive rings and to the class of left and right quasi-reflexive rings (semigroups with 0), respectively. Having done that we then turn our attention to minimal (0-minimal) quasi-ideals [cf. § 2]. As a by-product, we use left and right quasi-reflexive rings to deal with a problem posed by L. Marki (cf. end of § 2).

In this note the term *ring* means associative ring (not necessarily with identity). A ring R will be called *right duo* if every right ideal is two-sided. Ideal without modifier will mean two sided ideal. A subgroup Q of $(R, +)$ is called a *quasi-ideal* of the ring R if $QR\cap RQ\subseteq Q$. A non-empty subset Q of a semigroup S is called a *quasi-ideal* of S if $QS\cap SQ\subseteq Q$. We shall call a non-zero quasi-ideal of a ring (semigroup with 0) *minimal* (*0-minimal*) if it does not properly contain any non-zero quasi-ideal. Following O. Steinfeld in [2] we say that a quasi-ideal Q of a ring R (semigroup with 0) is *canonical* if Q is the intersection of a minimal (0-minimal) right ideal K and a minimal (0-minimal) left ideal L i.e. $0\neq Q=K\cap L$. Finally A^* will denote the right ideal $\{a-ea \mid a\in A\}$ of ring R where A is a given right ideal and e a central idempotent in R.

1. Central idempotents and right quasi-reflexive ideals.

Proposition 1 (cf. [1], Theorem 2.1). *Let R be a right quasi-reflexive ring and e an idempotent in R. The following are equivalent.*

a) *eR is a right quasi-reflexive ideal of R.*

b) *eR is an ideal R.*

c) *e is central in R.*

Proof. a)→b) is obvious. To show b)→c), note that the left annihilator of eR coincides with the right one of eR. Thence $ese=es$ for all $s\in R$.