91. A Holomorphic Structure of the Arithmetic-geometric Mean of Gauss

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§1. **Introduction.** For a, $b>0$, we define two sequences $\{a_n\}$ and $\{b_n\}$ by

$$\begin{aligned} &a_0=a, && b_0=b \\ &a_{n+1}=\frac{1}{2}(a_n+b_n), && b_{n+1}=\sqrt{a_n b_n}, \quad n=0,1,2,\cdots. \end{aligned} \tag{1.1}$$

It is well known and easily proved that both sequences converge to a common limit

$$M(a,b)=\lim a_n=\lim b_n,$$

which is called the arithmetic-geometric mean of a and b.

When a and b are complex numbers, we can define a sequence $\{(a_n, b_n)\}$ by the same algorithm (1.1). However, since there are two choices for b_{n+1} at each step of (1.1), we get uncountably many sequences $\{(a_n, b_n)\}$, which make the situation much more complicated than in the real case. Although the study of this case was initiated by Gauss, we refer to Cox [1, 2] as a modern account of what happens to the arithmetic-geometric mean of two complex numbers.

We assume

$$(\mathrm{A}) \qquad a, b\in \boldsymbol{C}, \quad ab\neq 0 \quad \text{and} \quad a\pm b\neq 0.$$

The excluded cases, though trivial, will turn out to be singular in a certain sense. It is easy to see that a_n and b_n also satisfy (A) for all $n\geq 0$.

A pair (a_n, b_n) is called *the right choice* if

$$\mathrm{Re}(b_n/a_n)>0 \quad \text{or} \quad \mathrm{Re}(b_n/a_n)=0, \quad \mathrm{Im}(b_n/a_n)>0.$$

Note that one of (a_n, b_n) and $(a_n, -b_n)$ is always the right choice, while the other is "the wrong choice".

One can prove that for any sequence $\{(a_n, b_n)\}$ the limit $\tau=\lim a_n=\lim b_n$ exists and that $\tau\neq 0$ if and only if all but finitely many of (a_n, b_n) are right choices ([1], [3]). Let $\mathfrak{M}(a,b)$ denote the set of such non-zero limits and $M(a,b)$ denote the limit attained by $\{(a_n, b_n)\}$ where (a_n, b_n) is the right choice for all $n\geq 1$.

Theorem (Cox [1], Geppert [4]). *Let a and b satisfy* (A). *Then all the values τ of $\mathfrak{M}(a,b)$ are given by*

$$\tau^{-1}=pM(a,b)^{-1}+iqM(a+b,a-b)^{-1},$$

where p and q are arbitrary relatively prime integers satisfying $p\equiv 1 \bmod 4$ and $q\equiv 0 \bmod 4$.

The purpose of this note is to give a sketch of a proof different from