84. *Zeta Zeros and Dirichlet L-functions.* II

By Akio FUJII
Department of Mathematics, Rikkyo University

(Communicated by Shokichi IYANAGA, M. J. A., Oct. 12, 1988)

We shall extend the investigations in [2] further. Let γ run over the positive imaginary parts of the zeros of the Riemann zeta function $\zeta(s)$. We are concerned with the distribution of $b(\gamma/2\pi)\log(\gamma/2\pi e\alpha)$ mod one. When $b>1$, the problem seems to be very difficult and our knowledge seems to be very scarce except our Theorem 5 below and a simple consequence of theorem in [1] with the help of Pjateckii-Sapiro's theorem in [4]. In this article we shall show that even the case for $0<b\leqq 1$ involves also the difficulty which lies as deep as the Generalized Riemann Hypothesis (G.R.H.) for Dirichlet L-functions $L(s,\chi)$. We assume the Riemann Hypothesis below.

We start with recalling the following fundamental theorem which is a special case of our main theorem in [1].

Theorem 1. *Let K be an integer $\geqq 1$ and let $T>T_0$. Then for any positive α,*

$$\sum_{\gamma<T} e\left(\frac{\gamma}{2\pi K}\log\left(\frac{\gamma}{2\pi e\alpha K}\right)\right) = -e^{(1/4)\pi i}\sqrt{\alpha}\,K \sum_{n<(T/2\pi\alpha K)^{1/K}} \Lambda(n)e(-\alpha n^K)n^{(1/2)(K-1)} + 0(T^{(2/5)+(1/2K)}(\log T\cdot\log\log T)^{4/5}) + 0(\sqrt{T}\log^2 T),$$

where we put $e(x)=e^{2\pi i x}$, $\Lambda(x)=\log p$ if $x=p^k$ with a prime number p and an integer $k\geqq 1$ and $\Lambda(x)=0$ otherwise.

When α is rational, we get the following corollary using the prime theorem in the arithmetic progressions.

Corollary 1. *Let K be an integer $\geqq 1$ and let $T>T_0$. Then for any integers a and $q\geqq 1$ with $(a,q)=1$, we have*

$$\sum_{\gamma<T} e\left(\frac{\gamma}{2\pi K}\log\left(\frac{\gamma}{2\pi e(a/q)K}\right)\right) = -e^{(1/4)\pi i}C\left(\frac{a}{q},\ K\right)(T/2\pi)^{(1/2)(1+(1/K))} + 0(T^{(1/2)(1+(1/K))}\exp(-C\sqrt{\log T})),$$

where we put $C(a/q,K)=2K^{(1/2)(1-(1/K))}\overline{S(a/q,K)}(K+1)^{-1}\varphi(q)^{-1}(a/q)^{-1/2K}$ and $S(a/q,K)=\sum'^{q}_{b=1}e((a/q)b^K)$, the dash indicates that b satisfies $(b,q)=1$, C denotes some positive constant and $\varphi(q)$ is the Euler function.

When α is irrational, using the estimate due to Vinogradov of $\sum_{n<Y}\Lambda(n)e(\alpha n^K)$ (cf. [6] and also Lemma 2 in [3]), we get the following corollary to Theorem 1 and Corollary 1, which has been mentioned only for the case for $K=1$ (cf. Corollary 5 in [1]).

Corollary 2. *Let K be an integer $\geqq 1$. Then we have*

$$\lim_{T\to\infty}(T/2\pi)^{-(1/2)(1+(1/K))}\sum_{\gamma<T} e\left(\frac{\gamma}{2\pi K}\log\left(\frac{\gamma}{2\pi e\alpha K}\right)\right) = \begin{cases} -e^{(1/4)\pi i}C(a/q,K) & \text{if } \alpha=a/q \text{ with integers } a \text{ and } q\geqq 1 \text{ and } (a,q)=1 \\ 0 & \text{if } \alpha \text{ is irrational.}\end{cases}$$