70. On Complexes in a Finite Abelian Group. I

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Let G be a finite abelian group, written additively, which will be fixed throughout this paper. We shall consider *complexes*, i.e. nonempty subsets of G. For $g \in G$ and two complexes A, B, we shall write

$$A+g=\{a+g \mid a \in A\},$$
$$A+B=\{a+b \mid a \in A,\ b \in B\},$$
$$A \circ B=\{a+b \mid a \in A,\ b \in B,\ a \neq b\}.$$

We were led to this latter operation $\circ$ in our geometric research [2] (see also [1]) on ovals in a finite projective plane, in which some of the results of the present paper were needed.

For a complex K, we shall write $|K|=k$, $|K \circ K|=m$. The object of this paper is to prove the following three theorems.

Theorem 1. *If $k=m>4$ for a complex K, then one of the two statements holds:*

(i) *K is a coset of a subgroup of G.*

(ii) *There exists an element g of G such that $K+g$ has only involutions and $(K+g) \cup \{0\}$ is a subgroup of G.*

Theorem 2. *If $K+K=K \circ K$, then one of the two statements holds:*

(i) *$K+K=K \circ K$ is a coset of a subgroup of G.*

(ii) $m \geq \frac{3}{2}k$.

Theorem 3. *If $|G|$ is odd and $K+K \neq K \circ K$, then*

$$m \geq \frac{k-3-\sqrt{5k^2-10k+9}}{2}=\frac{\sqrt{5}+1}{2}k-\frac{\sqrt{5}+3}{2}+0\left(\frac{1}{k}\right).$$

Let us begin with

Lemma 1. *K is a coset of a subgroup of G if and only if $k=|K+K|$.*

Proof. The only if part is obvious. Suppose now $k=|K+K|$. Let $a \in K$ and put $K-a=H$. Then clearly $0 \in H$, $|H|=k$ and $|H+H|=|K+K|=|H|$ and $H \subset H+H$ which implies $H=H+H$. Thus H is a subgroup of G and K is a coset of H.

Corollary. *If $k=m$ and $K \circ K=K+K$, then K is a coset of a subgroup of G.*

This corollary allows us to reformulate our Theorem 1 into the following form.

Theorem 1′. *Suppose $K+K \neq K \circ K$ and $k=m>4$. Then* (ii) *of Theo-*

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