# 58. A Mathematical Theory of Randomized Computation. III

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Based on the results of earlier notes [5], we shall show that the category of randomized domains forms a cartesian closed monoid, which yields c.c.m. reduction calculi equivalent to type-free $\lambda$-calculi [1]. Then we shall axiomatize randomized domains, and show that our randomized domain theory is a natural probabilistic extension of Scott's theory. We also construct the reflexive graph model $\mathcal{G}\omega$ similar to Scott's $\mathcal{P}\omega$ [3].

**11. The universal randomized domain $\mathcal{R}_\infty$.** A reflexive object $\mathcal{R}_\infty$ in the c.c.c. ***CBL*** is constructed in quite the same way with the construction of $\boldsymbol{D}_\infty$ in Scott's theory [2]: Let $\mathcal{R}_0 := \mathcal{R}$ be any nontrivial domain in ***CBL*** and $\mathcal{R}_{n+1} := [\mathcal{R}_n \to \mathcal{R}_n]$ ($\forall n \geq 0$). Define $\varphi_n : \mathcal{R}_n \to \mathcal{R}_{n+1}$ and $\psi_n : \mathcal{R}_{n+1} \to \mathcal{R}_n$ by $\varphi_0(x) := \lambda y \in \mathcal{R}_0 \cdot x$ ($\forall x \in \mathcal{R}_0$), $\psi_0(y) := y(0)$ ($\forall y \in \mathcal{R}_1$), $\varphi_{n+1}(x) := \varphi_n \circ x \circ \psi_n$ ($\forall x \in \mathcal{R}_{n+1}$), and $\psi_{n+1}(y) := \psi_n \circ y \circ \varphi_n$ ($\forall y \in \mathcal{R}_{n+2}$), for $\forall n \geq 0$.

Then $\langle \mathcal{R}_n, \psi_n \rangle_{n \in N}$ is a projective system of the domains in ***CBL***.

Let $x_n$ denote the $n$-th coordinate of $x = (x_n)_{n=0}^\infty$ of the product $\prod_{n=0}^\infty \mathcal{R}_n$. Define the projective limit $\mathcal{R}_\infty$ by $\mathcal{R}_\infty := \varprojlim \langle \mathcal{R}_n, \psi_n \rangle = \{x \in \prod_{n=0}^\infty \mathcal{R}_n | \forall n \in \boldsymbol{N}, \psi_{n+1}(x_{n+1}) = x_n\}$. Then $\mathcal{R}_\infty \in \boldsymbol{CBL}$ by (15)–(16). Define evaluation $\cdot$ in $\mathcal{R}_\infty$ by $x \cdot y := \sup_n x_{n+1}(y_n)$. Then the evaluation $\cdot$ on $\mathcal{R}_\infty$ is positive order continuous. And we have:

(25) (i) (Extensionality) (a) $x \leq y \to \forall z \in \mathcal{R}_\infty, x \cdot z \leq y \cdot z$. (b) $x = y \to \forall z \in \mathcal{R}_\infty, x \cdot z = y \cdot z$. (ii) (Comprehension) *Define for* $f \in [\mathcal{R}_\infty \to \mathcal{R}_\infty]$, $\square f := \sup_n \{\lambda y \in \mathcal{R}_n \cdot (f(y))_n\}$. *Then for* $\forall y \in \mathcal{R}_\infty$, $f(y) = \square f \cdot y$. (iii) (Reflexivity) $\mathcal{R}_\infty = [\mathcal{R}_\infty \to \mathcal{R}_\infty]$ *up to order isomorphism* (*and homeomorphism in the norm topology*).

Now that we have constructed a reflexive domain $\mathcal{R}_\infty$, the constructions of universal domains are straightforward: In fact, let $X$ be the two point space of Boolean values and $\mathcal{R}_0 := \mathcal{H}(\ell^1(X))$ and construct $\mathcal{R}_\infty$ with this $\mathcal{R}_0$. Then we can define positive order continuous pairing function and associated selector functions. So $\mathcal{R}_\infty \times \mathcal{R}_\infty$ is a retract of $\mathcal{R}_\infty$ with these functions. Hence $\mathcal{R}_\infty$ is a *universal* domain of ***CBL*** and ***CBL*** forms a *cartesian closed monoid.*

The notion of band in our theory exactly corresponds to the notion of retract in Scott's theory. We recall the definitions:

(25) Let $V$ be a $BL$. (i) A set $A \subset V$ is *solid* if $x \in A$, $y \in V$ and $|y| \leq |x| \Rightarrow y \in A$. (ii) An *ideal* of $V$ is a solid vector subspace of $V$. (iii) An ideal $B$ of $V$ is a *band* of $V$ if for $\forall$ non-empty $S \subset B$ possessing a supre-