48. Classification of Normal Congruence Subgroups of $G(\sqrt{q})$. I

By Toshikazu TAKAGI
College of Arts and Sciences, Showa University

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1. Normal congruence subgroups of the modular group $SL_2(\boldsymbol{Z})$ was completely classified by McQuillan [1]. As a continuation, Parson [2] attempted the classification of normal congruence subgroups of the group $G(\sqrt{q})$ $(q=2, 3)$ and obtained partial results. The present author classified all normal congruence subgroups of the group $G(\sqrt{q})$ for any prime q ([3]). In this note, the main results of [3] are reported.

2. In the following, we denote by $(a, b; c, d)$ a 2×2 matrix such that the first (resp. second) row is $(a\ b)$ (resp. $(c\ d)$). For a rational prime q, the group $G(\sqrt{q})\,(=\Gamma)$ is defined by $G(\sqrt{q})=W^{-1}N(\Gamma_0(q))W$, where $W=(1, 0; 0, \sqrt{q})$ and $N(\Gamma_0(q))$ is the normalizer of $\Gamma_0(q)$ in $SL_2(\boldsymbol{R})$. The group $\Gamma^e=W^{-1}\Gamma_0(q)W$ is a normal subgroup of Γ with index 2, and $\Gamma=\Gamma^e\cup S\Gamma^e$ where $S=(0, -1; 1, 0)$. We call elements of Γ^e (resp. $S\Gamma^e$) *even* (resp. *odd*). Also a subgroup G is called *even* or *odd* according as $G\subset\Gamma^e$ or $G\not\subset\Gamma^e$. Let R and $\mathfrak{n}$ denote the ring of integers of the quadratic field $\boldsymbol{Q}(\sqrt{q})$ and a non-zero ideal of R respectively. Since Γ is a subgroup of $SL_2(R)$, the principal congruence subgroup $\Gamma(\mathfrak{n})$ of Γ can be defined as usual. Set $\boldsymbol{L}=\boldsymbol{N}\cup\boldsymbol{N}q^{1/2}$. A subgroup G of Γ is called a *congruence subgroup* if G contains $\Gamma^e(L)$ $(=\Gamma(L)\cap\Gamma^e)$ for some $L\in\boldsymbol{L}$, and the *level* of G is defined to be the smallest element L with such a property. We shall classify in § 3–4 (resp. 5) even (resp. odd) normal congruence subgroups.

3. For each $L\in\boldsymbol{L}$, set $H_q(L)=\Gamma^e/\Gamma^e(L)$. For a subgroup N of $H_q(L)$, the *level* of N can be defined similarly as in case of a subgroup of Γ. Denote by σ the automorphism of Γ^e defined by $X\mapsto S^{-1}XS$. σ induces an automorphism of $H_q(L)$, which is also denoted by σ. Then in order to classify all even normal congruence subgroups, it is sufficient to classify all normal σ-subgroups of $H_q(L)$ which are of level L.

Here we treat the case where L is a power of a prime. Suppose now that $L=q^s$ with $q\neq 2$, where $s=m$ or $m-1/2$ $(m\in\boldsymbol{N})$. Since $H_q(q^{1/2})$ is a cyclic group of order $q-1$, there exists a unique subgroup of $H_q(q^{1/2})$ of index ν for each divisor ν of $q-1$ $(\nu\neq 1)$. It is denoted by $T^{(q)}_{(\nu)}$. Let B_{m-1} and C_{m-1} be two elements of $H_q(q^m)$ defined by $B_{m-1}=(1, q^{m-1}\overline{\sqrt{q}}; 0, 1)$ and $C_{m-1}=(1, 0; q^{m-1}\overline{\sqrt{q}})$ where $\overline{}$ indicates residue class mod L. When $q=3$ or 5, we denote by $R^{(q)}_m$ (resp. $S^{(q)}_m$) the cyclic group of order q generated by $B_{m-1}C^{-1}_{m-1}$ (resp. $B_{m-1}C_{m-1}$).

Theorem 1. *When $L=q^s$ $(q\neq 2,\ s=m,\ m-1/2\ (m\in\boldsymbol{N}))$, all normal σ-*