46. Symmetrization of the van der Corput Generalized Sequences

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1. Introduction. Let $\sigma=(x_n)_0^\infty$ be an infinite sequence in the unit interval $E=[0,1]$. The sequence σ is called *uniformly distributed* in E if $\lim_{N\to\infty} A_N(\sigma;x)=x$ for all $x\in E$, where $A_N(\sigma;x)/N$ denotes the number of terms x_n, $0\leqq n\leqq N-1$, which are less than x. The *diaphony* $F_N(\sigma)$ and the *L^2 discrepancy* $T_N(\sigma)$ of the sequence σ are defined for every positive integer N as follows:

$$F_N(\sigma)=(2\textstyle\sum_{h=1}^\infty (1/h^2)\,|(1/N)S_N(\sigma;h)|^2)^{1/2}$$

and

$$T_N(\sigma)=\left(\int_0^1 |A_N(\sigma;x)/N-x|^2\,dx\right)^{1/2},$$

where

$$S_N(\sigma;h)=\textstyle\sum_{n=0}^{N-1}\exp(2\pi i h x_n)$$

is the exponential sum of σ. It is well known (see [9] and [10]), that both $T_N(\sigma)\to 0$ and $F_N(\sigma)\to 0$ are equivalent to the sequence σ being uniformly distributed in E. Also it is known (see [5] and [6]), that the best possible order of magnitude of both $T_N(\sigma)$ and $F_N(\sigma)$ is $N^{-1}(\log N)^{1/2}$.

Now let $(r_j)_1^\infty$ be a given infinite sequence of integers $r_j\geqq 2$. Suppose also that for every integer $j\geqq 0$ we are given a permutation τ_j of the set $\{0,1,\cdots,r_{j+1}-1\}$. For the sake of brevity, we put $R_0=0$ and $R_j=r_1r_2\cdots r_j$ for $j\geqq 1$. The *van der Corput generalized sequence* $\sigma=(\varphi(n))_0^\infty$, associated with the given sequences $(r_j)_1^\infty$ and $(\tau_j)_0^\infty$, was constructed by Faure [2] as follows: For an integer $n\geqq 0$, let

$$n=\textstyle\sum_{j=0}^\infty a_jR_j \qquad (a_j\in\{0,1,\cdots,r_{j+1}-1\},\ j=0,1,\cdots)$$

be the (r_j)-adic expansion of n. Then set

$$\varphi(n)=\textstyle\sum_{j=0}^\infty \tau_j(a_j)/R_{j+1}.$$

In the present paper, we prove that if the sequence $(r_j)_1^\infty$ satisfies the condition $\sum_{j=1}^n r_j^2=O(n)$, then both the diaphony $F_N(\sigma)$ of the van der Corput generalized sequence σ and the L^2 discrepancy $T_N(\tilde\sigma)$ of any symmetric sequence $\tilde\sigma$ produced by σ have the best possible order of magnitude $N^{-1}(\log N)^{1/2}$. Also we obtain an exact estimate for the L^2 discrepancy of a class of two-dimensional finite sequences associated with the van der Corput generalized sequences.

2. Statement of the results.

Theorem 1. *Suppose that $(r_j)_1^\infty$ satisfies the condition*

(1) $$\textstyle\sum_{j=1}^n r_j^2\leqq Bn \qquad \textit{for all } n\in \mathbf{N},$$