# 23. On Traces of Hecke Operators on the Spaces of Cusp Forms of Half-integral Weight

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The space of cusp forms of half-integral weight, besides the space of those of integral weight, and Hecke operators on these spaces are well-known objects of arithmetical study of automorphic forms. After S. Niwa [2] had given some explicit relations between traces of these operators, W. Kohnen [1] gave further relation of similar type. In this paper, we shall give more general relations including these results. Details will appear in [3].

**Preliminaries.**

(a) **General notations.** Let $k$ be a positive integer. If $z \in \boldsymbol{C}$ and $x \in \boldsymbol{C}$, we put $z^x = \exp(x \cdot \log(z))$ with $\log(z) = \log(|z|) + \sqrt{-1}\arg(z)$, $\arg(z)$ being determined by $-\pi < \arg(z) \leq \pi$. For $z \in \boldsymbol{C}$, we put

$$e(z) = \exp(2\pi\sqrt{-1}\,z).$$

Let $\mathfrak{H}$ be the complex upper half plane. For a complex-valued function $f(z)$ on $\mathfrak{H}$, $\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2^+(\boldsymbol{R})$, $\gamma = \begin{pmatrix} u & v \\ w & x \end{pmatrix} \in \Gamma_0(4)$ and $z \in \mathfrak{H}$, we define functions $J(\alpha, z)$, $j(\gamma, z)$ and $f|[\alpha]_k(z)$ on $\mathfrak{H}$ by: $J(\alpha, z) = cz + d$, $j(\gamma, z) = \left(\frac{-1}{x}\right)^{-1/2} \times \left(\frac{w}{x}\right)(wz + x)^{1/2}$ and $f|[\alpha]_k(z) = (\det\alpha)^{k/2} J(\alpha, z)^{-k} f(\alpha z)$.

(b) **Modular forms of integral weight.** Let $N$ be a positive integer. By $S(2k, N)$, we denote the space of all holomorphic cusp forms of weight $2k$ with the trivial character on the group $\Gamma = \Gamma_0(N)$.

Let $\alpha \in GL_2^+(\boldsymbol{R})$. If $\Gamma$ and $\alpha^{-1}\Gamma\alpha$ are commensurable, we define a linear operator $[\Gamma\alpha\Gamma]_{2k}$ on $S(2k, N)$ by:

$$f|[\Gamma\alpha\Gamma]_{2k} = (\det\alpha)^{k-1} \sum\nolimits_{\alpha_i} f|[\alpha_i]_{2k},$$

where $\alpha_i$ runs over a system of representatives for $\Gamma\backslash\Gamma\alpha\Gamma$.

For a natural number $n$ with $(n, N) = 1$, we put

$$T_{2k,N}(n) = \sum\nolimits_{ad=n} \left[\Gamma \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \Gamma\right]_{2k},$$

where the sum is extended over all pairs of integers $(a, d)$ such that $a, d > 0$, $a|d$ and $ad = n$. Moreover, let $L_0$ be a positive divisor of $N$ such that $(L_0, N/L_0) = 1$ and that $L_0 \neq 1$. Take any element $\gamma(L_0) \in SL_2(\boldsymbol{Z})$ which satisfies the conditions: