22. On the Rank of the Elliptic Curve $y^2=x^3+k$

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(Communicated by Shokichi IYANAGA, M. J. A., March 12, 1987)

Let F be a finitely generated field over a prime field and $k\in F$. The F-points of the elliptic curve

$$E(k): y^2=x^3+k$$

form a finitely generated abelian group with respect to the well-known addition on $E(k)$. The rank of this group will be also called the rank of the curve $E(k)$ and denoted by $r(k)$. In this note, we consider the case $F=Q(p, q)$ where p, q are variables and give an example of the elliptic curve $E(k)$ with $r(k)\geqq 5$.

Let us first consider the case with the field F in general, and suppose $a, b, c, d\in F$. In our previous note [3], we showed that $E(k)$ with

$$k=(a^6+b^6+c^6-2a^3b^3-2b^3c^3-2c^3a^3)/4 \tag{1}$$

has 5 F-points $P_i=(x_i, y_i)$ $(i=1, \cdots, 5)$

$$\begin{aligned} x_1&=ab & y_1&=(a^3+b^3-c^3)/2\\ x_2&=ac & y_2&=(a^3-b^3+c^3)/2\\ x_3&=bc & y_3&=(-a^3+b^3+c^3)/2\\ x_4&=bd & y_4&=(-d^3-b^3+c^3)/2\\ x_5&=cd & y_5&=(-d^3+b^3-c^3)/2, \end{aligned} \tag{2}$$

provided that

$$a^3+d^3=2(b^3+c^3). \tag{3}$$

In [3], we utilized the parametric solution

$$\begin{aligned} a&=72t^4\\ b&=36t^3-1\\ c&=1\\ d&=-72t^4+6t \end{aligned} \tag{4}$$

of (3) to show that there are infinitely many values of $t\in Z$, for which $E(k)$ has at least 20 coprime Z-points.

Observe now that (3) has the following parametric solution

$$\begin{aligned} a&=-2p-2q+8(p^2-pq+q^2)^2\\ b&=-1+4(p-2q)(p^2-pq+q^2)\\ c&=1-4(p+q)(p^2-pq+q^2)\\ d&=2p-4q-8(p^2-pq+q^2)^2 \end{aligned} \tag{5}$$

(cf. Hardy and Wright [2] p. 199). This solution gives (4) as a specialization $p=t$, $q=-t$.

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