91. *On a Conjecture of Ono on Real Quadratic Fields*

By Ming-Guang Leu
Department of Mathematics, The Johns Hopkins University

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Let k be a quadratic field. We shall denote by Δ_k, M_k, h_k and χ_k, the discriminant, the Minkowski constant, the class number and the Kronecker character, respectively. Consider the following set

$$S_k=\{p, \text{ rational prime}\,;\ p\leqq M_k,\ \chi_k(p)\neq -1\}.$$

It is easy to see that the ideal class group H_k of k is generated by the classes of prime ideals $\mathfrak{p}$, $\mathfrak{p}\,|\,p$, $p\in S_k$. In particular, we have

(1) $$S_k=\phi \Longrightarrow h_k=1.$$

If k is *imaginary*, it is easy to prove the stronger relation:

(2) $$S_k=\phi \Longleftrightarrow h_k=1.$$ [1)]

However, if k is *real*, (2) is not always true; e.g. $h_k=1$ but $S_k=\{2\}$ for $k=\boldsymbol{Q}(\sqrt{6})$. In view of the celebrated conjecture of Gauss on the class number of real quadratic fields, it is interesting to determine all k's such that $S_k=\phi$. Recently Prof. Ono conjectured that these must be exactly the following 11 fields $k=\boldsymbol{Q}(\sqrt{m})$ with $m=2, 3, 5, 13, 21, 29, 53, 77, 173, 293$ and 437.

In this paper, we shall prove the following:

Theorem. *There are at most* 12 *real quadratic fields k such that* $S_k=\phi$.

In the sequel, m will denote a square-free natural number $\geqq 5$ and $h(m)$ the class number h_k with $k=\boldsymbol{Q}(\sqrt{m})$.[2)] We remind the reader that $M_k=\sqrt{\Delta_k}/2$ and that

$$\chi_k(p)=\begin{cases}\left(\dfrac{\Delta_k}{p}\right) & \text{if } p\neq 2,\ p\nmid \Delta_k,\\ (-1)^{(\Delta_k^2-1)/8} & \text{if } p=2,\ 2\nmid \Delta_k,\\ 0 & \text{if } p\,|\,\Delta_k.\end{cases}$$

Since $\chi_k(2)=0$ for $m\equiv 2, 3 \pmod 4$ and $\chi_k(2)=1$ for $m\equiv 1 \pmod 8$, S_k contains 2, i.e. $S_k\neq\phi$. *So from now on we can assume that* $m\equiv 5 \pmod 8$. Under this assumption, we shall try to determine k for which $S_k=\phi$.

The theorem obviously follows from the following two Propositions (A), (B).

Proposition (A). *There exists at most one $m\geqq e^{16}$ with $S_k=\phi$.*

Proposition (B). *If $S_k=\phi$ and $m<e^{16}$, then* $m=2, 3, 5, 13, 21, 29, 53, 77, 173, 293, 437$.

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1) The relation (2) is independent of deep results such as [1], [3].

2) Clearly, $S_k=\phi$ for $m=2$ or 3.